Equation of motion for relativistic compact binaries with the strong field point particle limit: Third post-Newtonian order

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An equation of motion for relativistic compact binaries is derived through the third post-Newtonian (3PN) approximation of general relativity. The strong field point particle limit and multipole expansion of the stars are used to solve iteratively the harmonically relaxed Einstein equations. We take into account the Lorentz contraction on the multipole moments defined in our previous works. We then derive a 3PN acceleration of the binary orbital motion of the two spherical compact stars based on a surface integral approach which is a direct consequence of local energy momentum conservation. Our resulting equation of motion admits a conserved energy (neglecting the 2.5PN radiation reaction effect), is Lorentz invariant, and is unambiguous: there exist no undetermined parameters reported in the previous works. We shall show that our 3PN equation of motion agrees physically with the Blanchet and Faye 3PN equation of motion if $\lambda = -1987/3080$, where λ is the parameter which is undetermined within their framework. This value of λ is consistent with the result of Damour, Jaranowski, and Schäfer, who first completed a 3PN iteration of the ADM Hamiltonian in the ADMTT gauge using dimensional regularization.

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I. INTRODUCTION

Tremendous effort has been made for direct detection of gravitational waves. Interferometric gravitational wave detectors such as GEO600 [1], the Laser Interferometric Gravitational Wave Observatory(LIGO) [2], and TAMA300 [3] have been operating successfully. They have been actively investigating the data and reported results [4–9].

One promising source of gravitational waves for those detectors is a relativistic compact binary system in an inspiraling phase. The detectability and quality of measurements of astrophysical information of such gravitational wave sources rely on the accuracy of our theoretical knowledge about the waveforms. A high-order, say, third- or fourth-order, post-Newtonian equation of motion for an inspiraling compact binary is one of the necessary ingredients to construct and study such waveforms [10–13].

In addition to the purpose of making accurate enough templates, the high-order post-Newtonian approximation for an inspiraling compact binary is a fruitful tool, for example, to construct astrophysically realistic initial data for numerical simulations [14–17] and to estimate innermost circular orbits [18,19].

The post-Newtonian (PN) equation of motion for relativistic compact binaries, a fundamental tool employed for the above purposes, has been derived by various authors (see reviews, e.g., [20–22]). The equation of motion for a two point-masses binary in a harmonic coordinate up to 2.5PN order, at which the radiation reaction effect first appears, was derived by Damour and Deruelle [23,24] based on the post-Minkowskian approach [25]. These works used Dirac delta distributions to express the point-masses mathematically, therefore they inevitably resorted to a purely mathematical regularization procedure to deal with divergences arising due to the nonlinearity of general relativity. Damour [20] gave a physical argument known as the "dominant Schwarzschild condition" which supports the use of Dirac delta distributions and their regularization up to 2.5PN order.

Direct validations of the Damour and Deruelle 2.5PN equation of motion have been obtained by several works [26–30]. Grishchuk and Kopeikin [26] and Kopeikin [27] worked on extended bodies with weak internal gravity. On the other hand, the present author, Futamase, and Asada derived the 2.5PN equation of motion [29] using the surface integral approach [31] and the strong field point-particle limit [32]. All the works quoted above agree with each other. Namely, our work [29] shows the applicability of the Damour and Deruelle 2.5PN equation of motion to a relativistic compact binary consisting of regular stars with strong internal gravity. We mention here that stars consisting of relativistic compact binaries, for which we are searching as gravitational wave sources, have strong internal gravitational field, and that it is a nontrivial question whether a star follows the same orbit regardless of the strength of its internal gravity.

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A 3PN iteration was first reported by Jaranowski and Schäfer [33]. There a 3PN Arnowitt-Deser-Misner (ADM) Hamiltonian in the ADM Transverse Traceless (ADMTT) gauge for two point-masses expressed as two Dirac delta distributions was derived based on the ADM canonical approach [34,35]. However, it was found in [33] that the regularizations they had used caused one coefficient $\omega_{\rm kinetic}$ undetermined in their framework. Moreover, they later found another undetermined coefficient in their Hamiltonian, called $\omega_{\rm static}$ [36]. Origins of these two coefficients were attributed to some unsatisfactory features of regularizations they had used, such as violation of Leibniz's rule. The former coefficient was then fixed as $\omega_{\rm kinetic} = 41/24$ by *a posteriori* imposing Poincaré invariance on their 3PN Hamiltonian [37]. As for the latter coefficient, Damour *et al.* [38] succeeded in fixing it as $\omega_{\rm static} = 0$, adopting dimensional regularization. Moreover, with this method they found the same value of $\omega_{\rm kinetic}$ as in [37], which ensures Lorentz invariance of their Hamiltonian.

On the other hand, Blanchet and Faye have succeeded in deriving a 3PN equation of motion for two point-masses expressed as two Dirac delta distributions in a harmonic coordinate [39,40] based on their previous work [28]. In this approach, they have assumed that two point-masses follow regularized geodesic equations (more precisely, they have assumed that the dynamics of two point-masses is described by a regularized action, from which the regularized geodesic equations were shown to be derived). The divergences due to their use of Dirac delta distributions were systematically regularized with the help of generalized Hadamard's partie finie regularization, which they have devised [41] and carefully elaborated [42] so that it respects Lorentz invariance. They found, however, that there exists one and only one undetermined coefficient (which they call λ). (But see [43] for the recent achievement of Blanchet and his collaborators.)

Interestingly, the two groups independently constructed a transformation between the two gauges and found that these two results coincide with each other when there exists a relationship [44,45]

$$\omega_{\rm static} = -\frac{11}{3}\lambda - \frac{1987}{840}. \tag{1.1}$$

Therefore, it is possible to fix the λ parameter from the result of [38]. However, applicability of mathematical regularizations to the current problem is not a trivial issue, but an assumption to be verified, or at least supported by convincing arguments. There is no argument such as the "dominant Schwarzschild condition" at the 3PN order. Thus, it seems crucial to achieve unambiguous 3PN iteration without introducing singular sources and to support (or give counterevidence against) the result of [38].

In this paper, we derive a third post-Newtonian equation of motion in a harmonic coordinate applicable to inspiraling binaries consisting of regular spherically symmetric compact stars with strong internal gravity, using yet another method which is based on our previous papers ([46] and [29], referred to as paper I and paper II, respectively, henceforth). Namely, to treat strong internal gravity of the stars, we have used the strong field point-particle limit and surface integral approach. This point-particle limit enables us to incorporate a notion of a self-gravitating point particle into general relativity without introducing singular sources.

In [47], we made a short report on our results. In this paper, we shall present the full explanation of our method and show our results.

The organization of this paper is as follows. In Secs. II and III, we briefly explain the basics of our method (see papers I and II for more details). After a short explanation on the structure of the 3PN equation of motion, we then compute the 3PN gravitational field required to derive the 3PN mass-energy relation from Secs. IV to VI. The 3PN mass-energy relation and the 3PN momentum-velocity relation are then derived in Secs. VII and VIII. Section IX describes how we evaluate the 3PN gravitational field necessary to derive a 3PN equation of motion. In these sections, we write down the formulas we have used and show several examples of our computations only, since the intermediate results are generally too lengthy to write down. We then show a 3PN equation of motion we obtained in Secs. X and XI and we shall compare it with the Blanchet and Faye 3PN equation of motion in Sec. XII 1. There, we found that our result is physically consistent with their result with $\lambda = -1987/3080$. Section XII 2 is devoted to a summary of our formalism and discussion. Some useful formulas and supplementary computations are shown in Appendixes.

Throughout, we use units where $c = 1 = G$. A tensor having alphabetical indexes, such as x^i or $\vec{x}$, denotes a Euclidean three-vector. We raise or lower its indexes with a Kronecker delta. An object having Greek indexes, such as x^μ, is a four-vector. Its indexes are raised or lowered by a flat Minkowskian metric.

II. STRONG FIELD POINT-PARTICLE LIMIT AND SCALINGS

In this section, we briefly review the strong field point-particle limit and associated scalings of the matter on the initial hypersurface. See [29,32,46,48–52] for more details.

We first introduce a nondimensional small parameter ϵ which represents slowness of a star's typical orbital velocity $\tilde{v}^i_{\rm orb}$ and thus which is a post-Newtonian expansion parameter,

$$\tilde{v}^i_{\rm orb} \equiv \frac{dx^i}{dt} \equiv \epsilon \frac{dx^i}{d\tau},$$

where we set $v^i \equiv dx^i/d\tau$ of order unity. We call τ the *Newtonian dynamical time* [32]. τ is a natural dynamical time scale of the orbital motion [32,49].

Then the post-Newtonian scaling implies that the typical scale of the mass of the star scales as ϵ^2.

Henceforth, we call the (τ, x^i) coordinate the near zone coordinate.

A. Strong field point-particle limit

One would think that a point-particle limit may be achieved by setting the radius of the star to zero. In general relativity, however, by this procedure the star cannot become a "point-particle", rather it becomes an extended body (black hole) whose radius is of order of its gravitational radius. One solution to avoid this conceptual problem was proposed by Futamase [32]. Following Futamase, we achieve a point-particle limit by letting the radius of the star shrink at the same rate as the mass of the star. This limit is nicely fit into the post-Newtonian approximation, since from the post-Newtonian scaling of the mass ($O(\epsilon^2)$), the radius of the star is $O(\epsilon^2) \to 0$ as we take the post-Newtonian limit ($\epsilon \to 0$). This point-particle has finite internal gravity since a typical scale of the self-gravitational field of the star, the mass over the radius, is finite. Thus we call this limit the *strong field point-particle limit.*

B. Surface integral approach and body zone

We construct a sphere for each star, called the *body zone* B_A for the star A ($A = 1, 2$) [32], which surrounds each star and does not overlap the other. More specifically, the scalings of the radius and the mass of the star motivate us to introduce the body zone of the star A, $B_A \equiv \{x^i || \vec{x} - \vec{z}_A(\tau)| < \epsilon R_A\}$ and a body zone coordinate of the star A, $\alpha^{\underline{i}}_A \equiv \epsilon^{-2}(x^i - z^i_A(\tau))$. Here $z^i_A(\tau)$ is a representative point of the star A, e.g., the center of mass of the star A. R_A, called the body zone radius, is an arbitrary length scale (much smaller than the orbital separation while ϵR_A is larger than the radius of the star for any ϵ) and constant (i.e., $dR_A/d\tau = 0$). Using the body zone coordinate, the star does not shrink when $\epsilon \to 0$, while the boundary of the body zone goes to infinity. Thus, it is appropriate to define the star's characteristic quantities such as a mass using the body zone coordinate.

On the other hand, the body zone serves us as a surface ∂B_A, through which gravitational energy momentum flux flows and in turn it amounts to gravitational force exerting on the star A. Since the body zone boundary ∂B_A is far away from the surface of the star A, we can evaluate explicitly the gravitational energy momentum flux on ∂B_A with the post-Newtonian gravitational field. After computing the surface integrals, we make the body zone shrink to derive the equation of motion for the compact star.

Effects of the internal structures of the compact stars may be coded in, e.g., multipole moments of the stars. These moments in turn appear in the gravitational energy momentum flux in the surface integral approach and affect the orbital motion.

C. Scalings on initial hypersurface

Following the works [50–52], we take the initial value formulation to solve Einstein equations. As the initial data for matter variables and gravitational field, we take a set of nearly stationary solutions of the exact Einstein equations representing two widely separated fluid balls. We assume that these solutions are parametrized by ϵ and the matter and the field variables have the following scalings on the initial hypersurface.

The matter density scales as ϵ^{-4} (in the (t, x^i) coordinate), implied by the scalings of the mass and the radius of the star. The internal time scale of the star is assumed to be comparable to that of the binary orbital motion. We assume that the star is pressure-supported.

From these initial data we have the following scalings of the star A's stress energy tensor components in the body zone coordinate, $T^{\mu\nu}_A$: $T^{\tau\tau}_A = O(\epsilon^{-2})$, $T^{\tau\underline{i}}_A = O(\epsilon^{-4})$, $T^{\underline{ij}}_A = O(\epsilon^{-8})$ on the initial hypersurface. Here the underlined indexes mean that for any tensor S^i, $S^{\underline{i}} = \epsilon^{-2} S^i$. In paper I, we regard the star A's body zone coordinate as a Fermi normal coordinate of the star and we have transformed $T^{\mu\nu}_N$, the components of the stress energy tensor of the matter in the near zone coordinate, to $T^{\mu\nu}_A$ using a transformation from the near zone coordinate to the Fermi normal coordinate at 1PN order. It is difficult, however, to construct the Fermi normal coordinate at a high post-Newtonian

order. Therefore, we shall not use it. We simply assume that for $T_N^{\mu\nu}$ (or rather for $\Lambda_N^{\mu\nu}$, the source term of the harmonically relaxed Einstein equations), $T_N^{\tau\tau} = O(\epsilon^{-2}), T_N^{\tau i} = O(\epsilon^{-4})$, and $T_N^{ij} = O(\epsilon^{-8})$, as their leading scalings.

As for the field variables on the initial hypersurface, we simply make a reasonable assumption that the field is of 2.5PN order except for the field determined by the constraint equations. If we take random initial data for the field [50] supposed to be of 1PN order, they are irrelevant to the dynamics of the binary system up to the radiation reaction order [52]. Thus, we expect that the 2.5PN order free data of the gravitational field on the initial hypersurface do not affect the orbital motion up to 3PN order.

III. FORMULATION

A. Field equation

As discussed in the previous section, we shall express our equation of motion in terms of surface integrals over the body zone boundary where it is assumed that the metric slightly deviates from the flat (auxiliary) metric $\eta^{\mu\nu} = \mathrm{diag}(-\epsilon^2, 1, 1, 1)$ (in the near zone coordinate (τ, x^i)). We define a deviation field $h^{\mu\nu}$ as

$$h^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g} g^{\mu\nu}, \tag{3.1}$$

where g is the determinant of the metric. Indexes of $h^{\mu\nu}$ are raised or lowered by the flat metric.

Now we impose a harmonic coordinate condition on the metric $h^{\mu\nu}{}_{,\nu} = 0$, where the comma denotes a partial derivative. In the harmonic gauge, we can recast Einstein equations into a relaxed form,

$$\Box h^{\mu\nu} = -16\pi\Lambda^{\mu\nu}, \tag{3.2}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat d'Alembertian and

$$\Lambda^{\mu\nu} \equiv \Theta^{\mu\nu} + \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta}, \tag{3.3}$$

$$\Theta^{\mu\nu} \equiv (-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}), \tag{3.4}$$

$$\chi^{\mu\nu\alpha\beta} \equiv \frac{1}{16\pi}(h^{\alpha\nu}h^{\beta\mu} - h^{\alpha\beta}h^{\mu\nu}). \tag{3.5}$$

Here, $T^{\mu\nu}$ and $t_{LL}^{\mu\nu}$ denote the stress-energy tensor of the stars and the Landau-Lifshitz pseudotensor [53]. In consistency with the harmonic condition, a local energy momentum conservation law is expressed as

$$\Lambda^{\mu\nu}{}_{,\nu} = 0. \tag{3.6}$$

Note that $\chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta}$ itself is divergence-free.

Now we rewrite the relaxed Einstein equations into an integral form,

$$h^{\mu\nu}(\tau, x^i) = 4\int_{C(\tau,x^k)} d^3y \frac{\Lambda^{\mu\nu}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon)}{|\vec{x} - \vec{y}|} + h_H^{\mu\nu}(\tau, x^i), \tag{3.7}$$

where $C(\tau, x^k)$ means the past light cone emanating from the event (τ, x^k). $C(\tau, x^k)$ is truncated on the $\tau = 0$ initial hypersurface. $h_H^{\mu\nu}$ is a homogeneous solution of a homogeneous wave equation in flat spacetime. In this paper, we shall adopt the no-incoming radiation condition (see, e.g., [54]) by taking sufficiently large τ, i.e., $h_H^{\mu\nu} = 0$.

We solve the Einstein equations as follows. First we split the integration region into two zones: the near zone and the far zone.

The near zone is the neighborhood of the gravitational wave source where the wave character of the gravitational radiation is not manifest. In this paper, we define the near zone as a sphere centered at some fixed point, enclosing both of the stars, and having a radius $\mathcal{R}/\epsilon$, where $\mathcal{R}$ is arbitrary but larger than the binary separation and the gravitational wavelength. The scaling of the near zone radius is derived from the ϵ dependence of the wavelength of the gravitational radiation emitted by the binary due to its orbital motion. The center of the near zone sphere would be determined, if necessary, for example, to be the center of mass of the near zone. The outside of the near zone is the far zone where the retardation effect of the field is crucial.

We evaluated the integrals over the far zone which contribute to the near zone field where the stars reside using the *direct integration of the relaxed Einstein equations* (DIRE) method, which was initiated by Will and his collaborators [30,55,56]. DIRE directly and nicely fits into our formalism since it utilizes the relaxed Einstein equations in the

harmonic gauge. Although we do not show our explicit computation in this paper, we have followed the DIRE method and checked that the far zone contribution does not affect the equation of motion through 3PN order. In fact, Blanchet and Damour [57], and later Pati and Will [56], showed that the far zone contribution to the near zone field affects (physically) the orbital motion starting at 4PN order. Henceforth we shall focus our attention on the near zone contribution to the near zone field,

$$h^{\mu\nu}(\tau, x^i) = 4\int_N d^3y \frac{\Lambda_N^{\mu\nu}(\tau - \epsilon|\vec{x}-\vec{y}|, y^k;\epsilon)}{|\vec{x}-\vec{y}|}, \tag{3.8}$$

where N denotes the near zone and we attach the subscript N to $\Lambda^{\mu\nu}$ to clarify that they are quantities in the near zone.

B. Near zone contribution

For the near zone contribution, we first make retardation expansion and change the integral region to a $\tau = \text{const}$ spatial hypersurface

$$h^{\mu\nu} = 4\sum_{n=0} \frac{(-\epsilon)^n}{n!}\left(\frac{\partial}{\partial\tau}\right)^n \int_N d^3y|\vec{x}-\vec{y}|^{n-1}\Lambda_N^{\mu\nu}(\tau, y^k;\epsilon). \tag{3.9}$$

Note that the above integral depends on the arbitrary length $\mathcal{R}$ in general. The cancellation between the $\mathcal{R}$-dependent terms in the far zone contribution and those in the near zone contribution was shown by Pati and Will [56] through sufficient post-Newtonian order. Moreover, the findings in [56,57] make us expect that we can remove $\mathcal{R}$-dependent terms via a suitable gauge transformation and then take a formal limit of infinite $\mathcal{R}$ in an equation of motion up to the 3PN level. With this expectation in mind, in the following we shall omit the terms with negative powers of $\mathcal{R}$ ($\mathcal{R}^{-k}; k > 0$) which vanish in the $\mathcal{R}$ infinite limit, while we shall retain terms with positive powers of $\mathcal{R}$ ($\mathcal{R}^k; k > 0$) and logarithmic terms ($\ln\mathcal{R}$) and see as a good computational check that these $\mathcal{R}$-dependent terms can be gauged away from our final result. Another reason to keep logarithmic terms ($\ln\mathcal{R}$) is to make the arguments of all possible logarithmic terms nondimensional.

Second we split the integral into two parts: contribution from the body zone $B = B_1 \cup B_2$, and from elsewhere, N/B. Schematically we evaluate the following two types of integrals (we omit the indexes):

$$h = \sum_{n=0}\left(h_{Bn} + h_{N/Bn}\right), \tag{3.10}$$

$$h_{Bn} = 4\frac{(-\epsilon)^n}{n!}\left(\frac{\partial}{\partial\tau}\right)^n \epsilon^6 \sum_{A=1,2}\int_{B_A} d^3\alpha_A \frac{f(\tau, \vec{z}_A + \epsilon^2\vec{\alpha}_A)}{|\vec{r}_A - \epsilon^2\vec{\alpha}_A|^{1-n}}, \tag{3.11}$$

$$h_{N/Bn} = 4\frac{(-\epsilon)^n}{n!}\left(\frac{\partial}{\partial\tau}\right)^n \int_{N/B} d^3y \frac{f(\tau, \vec{y})}{|\vec{x}-\vec{y}|^{1-n}}, \tag{3.12}$$

where $\vec{r}_A \equiv \vec{x} - \vec{z}_A$ and $f(\tau, x^k)$ is some function. We shall deal with these two contributions successively in the following.

1. Body zone contribution

As for the body zone contribution, we make a multipole expansion being concerned with the scaling of the integrand, i.e., $\Lambda^{\mu\nu}$ in the body zone. For example, the $n = 0$ terms in Eq. (3.9), $h^{\mu\nu}_{Bn=0}$, give

$$
h^{\tau\tau}_{Bn=0} = 4\epsilon^4 \sum_{A=1,2} \left(\frac{P^\tau_A}{r_A} + \epsilon^2 \frac{D^k_A r^k_A}{r^3_A} + \epsilon^4 \frac{3I^{<kl>}_A r^k_A r^l_A}{2r^5_A} + \epsilon^6 \frac{5I^{<klm>}_A r^k_A r^l_A r^m_A}{2r^7_A} \right) + O(\epsilon^{12}), \tag{3.13}
$$

$$
h^{\tau i}_{Bn=0} = 4\epsilon^4 \sum_{A=1,2} \left(\frac{P^i_A}{r_A} + \epsilon^2 \frac{J^{ki}_A r^k_A}{r^3_A} + \epsilon^4 \frac{3J^{<kl>i}_A r^k_A r^l_A}{2r^5_A} \right) + O(\epsilon^{10}), \tag{3.14}
$$

$$
h^{ij}_{Bn=0} = 4\epsilon^2 \sum_{A=1,2} \left(\frac{Z^{ij}_A}{r_A} + \epsilon^2 \frac{Z^{kij}_A r^k_A}{r^3_A} + \epsilon^4 \frac{3Z^{<kl>ij}_A r^k_A r^l_A}{2r^5_A} + \epsilon^6 \frac{5Z^{<klm>ij}_A r^k_A r^l_A r^m_A}{2r^7_A} \right) + O(\epsilon^{10}), \tag{3.15}
$$

where $r_A \equiv |\vec{r}_A|$. The quantity with $<>$ denotes symmetric and trace-free (STF) on the indexes between the brackets. To derive the 3PN equation of motion, we need $h^{\tau\tau}$ up to $O(\epsilon^{10})$ and $h^{\tau i}$ and h^{ij} up to $O(\epsilon^8)$.

In the above equations we defined multipole moments of the star A as

$$
I^{K_l}_A \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda^{\tau\tau}_N \alpha^{K_l}_A, \tag{3.16}
$$

$$
J^{K_l i}_A \equiv \epsilon^4 \int_{B_A} d^3\alpha_A \Lambda^{\tau i}_N \alpha^{K_l}_A, \tag{3.17}
$$

$$
Z^{K_l ij}_A \equiv \epsilon^8 \int_{B_A} d^3\alpha_A \Lambda^{ij}_N \alpha^{K_l}_A, \tag{3.18}
$$

where the capital index denotes a set of collective indexes, $I_l \equiv i_1 i_2 \cdots i_l$ and $\alpha^{I_l}_A \equiv \alpha^{i_1}_A \alpha^{i_2}_A \cdots \alpha^{i_l}_A$. Then $P^\tau_A \equiv I^{I_0}_A$, $D^{i_1}_A \equiv I^{I_1}_A$, and $P^{i_1}_A \equiv J^{I_1}_A$. We simply call P^μ_A the four-momentum of the star A, P^i_A the (three-)momentum, and P^τ_A the energy. Also we call D^i_A the dipole moment and I^{ij}_A the quadrupole moment.

Then we transform these moments into more convenient forms using the conservation law Eq. (3.6). In the following, $v^i_A \equiv \dot{z}^i_A$, an overdot denotes τ time derivative, and $\vec{y}_A \equiv \vec{y} - \vec{z}_A$. Noticing that the body zone remains unchanged (in the near zone coordinate), i.e., $\dot{R}_A = 0$, we have

$$
P^i_A = P^\tau_A v^i_A + Q^i_A + \epsilon^2 \frac{dD^i_A}{d\tau}, \tag{3.19}
$$

$$
J^{ij}_A = \frac{1}{2}\left(M^{ij}_A + \epsilon^2 \frac{dI^{ij}_A}{d\tau} \right) + v^{(i}_A D^{j)}_A + \frac{1}{2}\epsilon^{-2} Q^{ij}_A, \tag{3.20}
$$

$$
Z^{ij}_A = \epsilon^2 P^\tau_A v^i_A v^j_A + \frac{1}{2}\epsilon^6 \frac{d^2 I^{ij}_A}{d\tau^2} + 2\epsilon^4 v^{(i}_A \frac{dD^{j)}_A}{d\tau} + \epsilon^4 \frac{dv^{(i}_A}{d\tau} D^{j)}_A + \epsilon^2 Q^{(i}_A v^{j)}_A + \epsilon^2 R^{(ij)}_A + \frac{1}{2}\epsilon^2 \frac{dQ^{ij}_A}{d\tau}, \tag{3.21}
$$

$$
Z^{kij}_A = \frac{3}{2} A^{kij}_A - A^{(ij)k}_A, \tag{3.22}
$$

where

$$
M^{ij}_A \equiv 2\epsilon^4 \int_{B_A} d^3\alpha_A \alpha^{[i}_A \Lambda^{j]\tau}_N, \tag{3.23}
$$

$$
Q^{K_l i}_A \equiv \epsilon^{-4} \oint_{\partial B_A} dS_m \left(\Lambda^{\tau m}_N - v^m_A \Lambda^{\tau\tau}_N \right) y^{K_l}_A y^i_A, \tag{3.24}
$$

$$
R^{K_l ij}_A \equiv \epsilon^{-4} \oint_{\partial B_A} dS_m \left(\Lambda^{mj}_N - v^m_A \Lambda^{\tau j}_N \right) y^{K_l}_A y^i_A, \tag{3.25}
$$

and

$$
A^{kij}_A \equiv \epsilon^2 J^{k(i}_A v^{j)}_A + \epsilon^2 v^k_A J^{(ij)}_A + R^{k(ij)}_A + \epsilon^4 \frac{dJ^{k(ij)}_A}{d\tau}. \tag{3.26}
$$

[ ] and ( ) denote antisymmetrization and symmetrization on the indexes between the brackets. M_A^{ij} is the spin of the star A and Eq. (3.19) gives a momentum-velocity relation. Thus our momentum-velocity relation is a direct analogue of the Newtonian momentum-velocity relation. In general, we have[1]

$$J_A^{K_l i} = J_A^{(K_l i)} + \frac{2l}{l+1} J_A^{(K_{l-1}[k_l)i]}, \tag{3.27}$$

$$Z_A^{K_l ij} = \frac{1}{2}\left[Z_A^{(K_l i)j} + \frac{2l}{l+1} Z_A^{(K_{l-1}[k_l)i]j} + Z_A^{(K_l j)i} + \frac{2l}{l+1} Z_A^{(K_{l-1}[k_l)j]i} \right], \tag{3.28}$$

and

$$J_A^{(K_l i)} = \frac{1}{l+1}\epsilon^2 \frac{dI_A^{K_l i}}{d\tau} + v_A^{(i} I_A^{K_l)} + \frac{1}{l+1}\epsilon^{-2l} Q_A^{K_l i}, \tag{3.29}$$

$$Z_A^{(K_l i)j} + Z_A^{(K_l j)i} = \epsilon^2 v_A^{(i} J_A^{K_l)j} + \epsilon^2 v_A^{(j} J_A^{K_l)i} + \frac{2}{l+1}\epsilon^4 \frac{dJ_A^{K_l(ij)}}{d\tau} + \frac{2}{l+1}\epsilon^{-2l+2} R_A^{K_l(ij)}. \tag{3.30}$$

The surface integrals $Q_A^{K_l i}$ and $R_A^{K_l ij}$ will be computed in Appendix A and do contribute to the field and the equation of motion starting at 3PN order.

2. N/B contribution

About the N/B contribution, since the integrand $\Lambda_N^{\mu\nu} = -g t_{LL}^{\mu\nu} + \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta}$ is at least quadratic in the small deviation field $h^{\mu\nu}$, we make the post-Newtonian expansion in the integrand. Then, basically, with the help of (super)potentials $g(\vec{x})$ which satisfy $\Delta g(\vec{x}) = f(\vec{x})$, Δ denoting a Laplacian, we have for each integral (e.g., the $n = 0$ term in Eq. (3.12))

$$\int_{N/B} d^3y \frac{f(\vec{y})}{|\vec{x}-\vec{y}|} = -4\pi g(\vec{x}) + \oint_{\partial(N/B)} dS_k \left[\frac{1}{|\vec{x}-\vec{y}|}\frac{\partial g(\vec{y})}{\partial y^k} - g(\vec{y})\frac{\partial}{\partial y^k}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\right]. \tag{3.31}$$

We show a derivation of Eq. (3.31) in Appendix B. For $n \geq 1$ terms in Eq. (3.12), we use (super)potentials many times to convert all the volume integrals into surface integrals and the bulk terms ("$-4\pi g(\vec{x})$"). [2]

Finding the superpotentials is one of the most formidable tasks, especially when we proceed to a high post-Newtonian order. Fortunately, up to 2.5PN order, all the required superpotentials are available [28,33]. At 3PN order, there appear many integrands for which we could not find the required superpotentials. To obtain a 3PN equation of motion, we use an alternative method. The details of the method will be explained later.

Finally, we note that $h^{\mu\nu} = O(\epsilon^4)$ as shown in paper II.

C. General form of the equation of motion

From the definition of the four-momentum,

$$P_A^\mu(\tau) \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda_N^{\tau\mu}, \tag{3.32}$$

and the conservation law, Eq. (3.6), we have an evolution equation for the four-momentum,

$$\frac{dP_A^\mu}{d\tau} = -\epsilon^{-4}\oint_{\partial B_A} dS_k \Lambda_N^{k\mu} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Lambda_N^{\tau\mu}. \tag{3.33}$$

---

[1] The equation in paper II corresponding to Eq. (3.30) has a misprint, though it does not affect the 2.5PN equation of motion.

[2] Notice that when solving a Poisson equation $\Delta g(\vec{x}) = f(\vec{x})$, a particular solution suffices for our purpose. By virtue of the surface integral term in Eq. (3.31), it is not necessary to be concerned about a homogeneous solution of the Poisson equation.

Here we used the fact that the size and the shape of the body zone are defined to be time-independent (in the near zone coordinate), while the center of the body zone moves at the velocity of the star's representative point.

Substituting the momentum-velocity relation, Eq. (3.19), into the spatial components of Eq. (3.33), we obtain the general form of the equation of motion for the star A,

$$\begin{aligned} P_A^\tau \frac{dv_A^i}{d\tau} &= -\epsilon^{-4} \oint_{\partial B_A} dS_k \Lambda_N^{ki} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Lambda_N^{\tau i} \\ &+ \epsilon^{-4} v_A^i \left(\oint_{\partial B_A} dS_k \Lambda_N^{k\tau} - v_A^k \oint_{\partial B_A} dS_k \Lambda_N^{\tau\tau} \right) \\ &- \frac{dQ_A^i}{d\tau} - \epsilon^2 \frac{d^2 D_A^i}{d\tau^2}. \end{aligned} \tag{3.34}$$

All the right-hand side terms in Eq. (3.34) except for the dipole moment are expressed as surface integrals. We can specify the value of D_A^i freely to determine the representative point $z_A^i(\tau)$ of the star A. For example, we may call $z_A^i(\tau)$ corresponding to $D_A^i = 0$ the center of mass of the star A from an analogy of the Newtonian dynamics.

In Eq. (3.34), P_A^τ rather than the mass of the star A appears. Hence we have to derive a relation between the mass and P_A^τ. We shall derive the relation by solving the temporal component of the evolution equation (3.33) functionally. In fact, at lowest order, we have shown in paper II that

$$\frac{dP_A^\tau}{d\tau} = O(\epsilon^2). \tag{3.35}$$

Then we define the mass of the star A as the integrating constant of this equation,

$$m_A \equiv \lim_{\epsilon \to 0} P_A^\tau. \tag{3.36}$$

m_A is the ADM mass that the star A would have if it were isolated. We took the ϵ zero limit in Eq. (3.36) to ensure that the mass defined above does not include the effect of the companion star and the orbital motion of the star itself. Some subtleties about this definition are discussed in paper II. By definition, m_A is constant. The procedure that we use to solve the evolution equation of P_A^τ and obtain the mass energy relation is achieved up to 3PN order successfully and the result will be shown later.

Since the general form of the equation of motion is expressed in terms of surface integrals over the body zone boundary, we can derive an equation of motion for a strongly self-gravitating star using the post-Newtonian approximation. Effects of the star's internal structure on the orbital motion such as tidally induced multipole moments appear through the field and hence the integrand $\Lambda_N^{\mu\nu}$ of the surface integrals.

D. Lorentz contraction and multipole moments

In this paper, we are concerned with a binary consisting of two spherically symmetric compact stars. In other words, all the multipole moments of the star defined in an appropriate reference coordinate where effects of its orbital motion and the companion star are removed (modulo, namely, the tidal effect) vanish. We adopt the generalized Fermi coordinate (GFC) [59] as a star's reference coordinate for this purpose

Then a question specific to our formalism is whether the differences between the multipole moments defined in Eqs. (3.16), (3.17), and (3.18) and the multipole moments in GFC give purely monopole terms. This problem is addressed in Appendix C and the differences are mainly attributed to the shape of the body zone. The body zone B_A which is spherical in the near zone coordinate (NZC) is not spherical in the GFC mainly because of a kinematic effect (Lorentz contraction). To derive a 3PN equation of motion, it is sufficient to compute the difference in the STF quadrupole moment up to 1PN order. The result is

$$\delta I_A^{<ij>} \equiv I_{A,\mathrm{NZC}}^{<ij>} - I_{A,\mathrm{GFC}}^{<ij>} = -\epsilon^2 \frac{4m_A^3}{5} v_A^{<i} v_A^{j>} + O(\epsilon^3), \tag{3.37}$$

where $I_{A,\mathrm{NZC}}^{ij} \equiv I_A^{ij}$. $I_{A,\mathrm{GFC}}^{ij}$ is the quadrupole moments defined in the generalized Fermi coordinate.

As is obvious from Eq. (3.37), this difference appears even if the companion star does not exist. We note that we could derive the 3PN metric for an isolated star A moving at a constant velocity using our method explained in this section by simply letting the mass of the companion star be zero. Actually, $\delta I_A^{<ij>}$ above is a necessary term which makes the so-obtained 3PN metric the same as the Schwarzschild metric boosted at the constant velocity $\vec{v}_A$ in the harmonic coordinate.

E. On the arbitrary constant R_A

Our final remark in this section is on the two arbitrary constants R_A. Since we introduce the body zones by hand, the arbitrary body zone radii R_A seem to appear in the metric, the multipole moments of the stars, and the equation of motion. As was discussed in detail in paper II, we proved that the surface integrals in Eq. (3.34) that we evaluate to derive the equation of motion do not depend on R_A through any order of the post-Newtonian iteration. As for the field and the multipole moments appearing in the field (mass, spin, quadrupoles, etc.), we reasonably expect that the R_A-dependent terms in the body zone contribution $h_B^{\mu\nu}$ and N/B contribution $h_{N/B}^{\mu\nu}$ cancel each other out, since the total field $h^{\mu\nu} = h_B^{\mu\nu} + h_{N/B}^{\mu\nu}$ is obviously independent of R_A. In the cancellation, the multipole moments should be "renormalized" so that those moments do not depend on R_A. Such cancellation among R_A-dependent terms and the "renormalization" was demonstrated explicitly in paper I up to 1PN order.

Practically, the above observation enables us to discard safely all the ϵR_A-dependent terms except for logarithms of ϵR_A. We retain $\ln \epsilon R_A$-dependent terms to nondimensionalize the arguments of the logarithms. Thus, instead of performing renormalization of the multipole moments, we simply discard the ϵR_A dependences other than the $\ln \epsilon R_A$ dependences. We emphasize here that we discard the ϵR_A-dependent terms in the field first and then evaluate the surface integrals in the general form of the equation of motion using the field which is independent of ϵR_A. We then discard the ϵR_A-dependent terms arising in the computation of the surface integrals.

Appendix D is devoted to an explanation on the renormalization of the stars' multipole moments. There, we also give a justification for our procedure of discarding through our computation of the field all the ϵR_A-dependent terms other than the $\ln \epsilon R_A$-dependent terms.

IV. STRUCTURE OF THE 3PN EQUATION OF MOTION

In the following sections, we shall derive an acceleration for two spherical compact stars through third post-Newtonian accuracy.

First, we split the four-momentum, the dipole moment, and the Q_A^i integral into two parts, namely the Θ part and the χ part.

$$P_{A\Theta}^{\mu} \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Theta_N^{\mu\tau}, \tag{4.1}$$

$$P_{A\chi}^{\mu} \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \chi_N^{\mu\tau\alpha\beta}{}_{,\alpha\beta}, \tag{4.2}$$

$$D_{A\Theta}^{i} \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \alpha_A^i \Theta_N^{\tau\tau}, \tag{4.3}$$

$$D_{A\chi}^{i} \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \alpha_A^i \chi_N^{\tau\tau\alpha\beta}{}_{,\alpha\beta}, \tag{4.4}$$

$$Q_{A\Theta}^{i} \equiv \epsilon^{-4} \oint_{\partial B_A} dS_k \left(\Theta_N^{\tau k} - v_A^k \Theta_N^{\tau\tau}\right) y_A^i, \tag{4.5}$$

$$Q_{A\chi}^{i} \equiv \epsilon^{-4} \oint_{\partial B_A} dS_k \left(\chi_N^{\tau k\alpha\beta}{}_{,\alpha\beta} - v_A^k \chi_N^{\tau\tau\alpha\beta}{}_{,\alpha\beta}\right) y_A^i. \tag{4.6}$$

Correspondingly, we split the momentum-velocity relation Eq. (3.19) and the evolution equation for the four-momentum Eq. (3.33) into the Θ part and the χ part.

Now, $P_{A\chi}^{\mu}$ and $D_{A\chi}^{i}$ as well as $Q_{A\chi}^{i}$ can be expressed completely as surface integrals, and can be computed explicitly into functions of m_A, $\vec{v}_A$, and $\vec{r}_{12}$. It is straightforward to compute them up to 3PN order, since we only need the 2PN field to perform the surface integrals. The results are shown in Appendix E. There, we also compute the χ part of the momentum-velocity relation and the evolution equation for $P_{A\chi}^{\mu}$. Then comparing these equations with $P_{A\chi}^{\mu}$, $D_{A\chi}^{i}$, and $Q_{A\chi}^{i}$, we found that the χ part of these equations is an identity up to 3PN order. This observation then means that the nontrivial momentum-velocity relation and the mass-energy relation, an equation of motion, are obtained from the Θ part of Eqs. (3.19) and (3.33).

Therefore, the equations that we have to evaluate to derive an evolution equation for the energy and an equation of motion are actually

$$\left(\frac{dP^\tau_{1\Theta}}{d\tau}\right)_{\le 3\mathrm{PN}} = \left(\frac{dP^\tau_{1\Theta}}{d\tau}\right)_{\le 2.5\mathrm{PN}} + \epsilon^6\left[-\oint_{\partial B_1} dS_{k10}\Theta_N^{\tau k} + v_1^k\oint_{\partial B_1} dS_{k10}\Theta_N^{\tau\tau}\right], \tag{4.7}$$

$$\begin{aligned} m_1\left(\frac{dv_1^i}{d\tau}\right)_{\le 3\mathrm{PN}} &= m_1\left(\frac{dv_1^i}{d\tau}\right)_{\le 2.5\mathrm{PN}} + \epsilon^6\left[-\oint_{\partial B_1} dS_{k10}\Theta_N^{ki} + v_1^k\oint_{\partial B_1} dS_{k10}\Theta_N^{\tau i}\right] \\ &+\epsilon^6\left(\frac{dP^\tau_{1\Theta}}{d\tau}\right)_{3\mathrm{PN}} v_1^i + \epsilon^6\left((m_1 - P^\tau_{1\Theta})\frac{dv_1^i}{d\tau}\right)_{3\mathrm{PN}} \\ &-\epsilon^6\frac{d_6Q^i_{1\Theta}}{d\tau} - \epsilon^6\frac{d^2{}_4D^i_{1\Theta}}{d\tau^2}, \end{aligned} \tag{4.8}$$

where for an equation or a quantity f, $(f)_{\le n\mathrm{PN}}$ and $(f)_{n\mathrm{PN}}$ denote f up to nPN order and f at nPN order, respectively. ${}_{\le n}f$ and ${}_nf$, on the other hand, denote an equation or a quantity f up to $O(\epsilon^n)$ and at $O(\epsilon^n)$. In paper II, we found $Q^i_{A\Theta} = O(\epsilon^6)$. For later convenience, in Eq. (4.8) we retain $D^i_{A\Theta}$ of order ϵ^4, which appears at a 3PN equation of motion (in other words, we set ${}_{\le 3}D^i_{A\Theta} = 0$). It should be understood that in the second line of Eq. (4.8), the acceleration $dv_1^i/d\tau$ should be replaced by the acceleration of an appropriate order lower than 2.5PN. We note that the χ parts of $Q_A^{K_l i}$ and $R_A^{K_l ij}$ integrals and the multipole moments including $P^\mu_{A\chi}$ and $D^i_{A\chi}$ affect an equation of motion through the field and hence the integrands of Eqs. (4.7) and (4.8). Henceforth, we call Eq. (4.8) the general form of the 3PN equation of motion.

The explicit forms of the integrands ${}_{10}\Theta_N^{\mu\nu} = {}_{10}[(-g)t^{\mu\nu}_{LL}]$ (on ∂B_A) are (see Appendix F),

$$16\pi_{10}\Theta_N^{\tau\tau} = -\frac{7}{4}{}_4h^{\tau\tau}{}_{,k8}h^{\tau\tau,k} + \cdots, \tag{4.9}$$

$$16\pi_{10}\Theta_N^{\tau i} = 2_4h^{\tau\tau}{}_{,k8}h^{\tau[k,i]} + \cdots, \tag{4.10}$$

$$\begin{aligned} 16\pi_{10}\Theta_N^{ij} &= \frac{1}{4}(\delta^i{}_k\delta^j{}_l + \delta^i{}_l\delta^j{}_k - \delta^{ij}\delta_{kl})\left\{{}_4h^{\tau\tau,k}({}_{10}h^{\tau\tau,l} + {}_8h^m{}_m{}^{,l} + 4_8h^{\tau l}{}_{,\tau}) + 8_4h^\tau{}_m{}^{,k}{}_8h^{\tau[l,m]}\right\} \\ &+ 2_4h^{\tau i}{}_{,k8}h^{\tau[k,j]} + 2_4h^{\tau j}{}_{,k8}h^{\tau[k,i]} + \cdots. \end{aligned} \tag{4.11}$$

The fields up to 2.5PN order, ${}_{\le 9}h^{\tau\tau}$, ${}_{\le 7}h^{\tau i}$, and ${}_{\le 7}h^{ij}$, are listed in paper II. Thus, to derive the 3PN mass-energy relation and the 3PN momentum-velocity relation, we have to derive ${}_8h^{\tau i}$. To derive the 3PN equation of motion, we further need ${}_{10}h^{\tau\tau} + {}_8h^k{}_k$.

Up to 2.5PN order, the superpotentials required to compute the field could be found [28,33]. However, at 3PN order, it is quite difficult to complete the required superpotentials. We take another method to overcome this problem. We shall explain our method in detail in Secs. V and IX. Below, we begin our calculation by deriving ${}_8h^{\tau i}$ and check (a part of) the 3PN harmonic condition ${}_{\le 8}h^{\tau\mu}{}_{,\mu} = 0$.

V. ${}_8H^{\tau I}$: N/B INTEGRALS

In this section, we derive ${}_8h^{\tau i}$,

$${}_{\le 8}h^{\tau i} = {}_{\le 8}h_B^{\tau i} + 4\epsilon^8\int_{N/B}\frac{d^3y_8\Lambda_N^{\tau i}}{|\vec{x}-\vec{y}|} + 2\epsilon^8\frac{\partial^2}{\partial\tau^2}\left[\int_{N/B} d^3y|\vec{x}-\vec{y}|_6\Lambda_N^{\tau i}\right], \tag{5.1}$$

where ${}_{\le 8}h_B^{\tau i}$ is the body zone contribution up to $O(\epsilon^8)$ and shown in Appendix A as Eq. (A9). The Q_A^{kli} integral contained in ${}_{\le 8}h_B^{\tau i}$ is found in Appendix A to vanish. We are concerned with the equation of motion for two spherical compact stars and we shall only retain monopole terms in ${}_{\le 8}h_B^{\tau i}$.

The second time derivative term in the retardation expansion (the last term in Eq. (5.1)) can be integrated explicitly via super-superpotentials (i.e., a particular solution of the Poisson equation with a superpotential as a source). The result is

$$\begin{aligned} \int_{N/B} d^3y|\vec{x}-\vec{y}|_6\Lambda_N^{\tau i} &= \frac{11P_1^\tau P_1^i}{3}\ln\left(\frac{r_1}{\mathcal{R}/\epsilon}\right) + \frac{11P_1^\tau P_1^i}{6} - \frac{P_1^\tau P_1^k}{12}n_1^{<ik>} + \frac{11P_1^\tau P_2^i}{6} \\ &+8P_1^\tau P_2^k\left(\delta^i{}_k\Delta_{12} - \partial_{z_1^k}\partial_{z_2^i} + \frac{3}{4}\partial_{z_1^i}\partial_{z_2^k}\right)f^{(\ln S)} \\ &-\frac{11P_1^\tau P_2^i}{3}\ln\left(\frac{2\mathcal{R}}{\epsilon}\right) + (1\leftrightarrow 2), \end{aligned} \tag{5.2}$$

where $\Delta_{AA'} \equiv \delta^{ij}\partial_{z^i_A}\partial_{z^j_{A'}}$ and $\partial_{z^i_A} = \partial/\partial z^i_A$. The symbol $(1 \leftrightarrow 2)$ denotes the same terms but with the star's label 1 exchanged for 2. $f^{(\ln S)}$ is a superpotential satisfying $\Delta f^{(\ln S)} = \ln S$, where $S \equiv r_1 + r_2 + r_{12}$, and its explicit expression is given in [33] as

$$f^{(\ln S)} = \frac{1}{36}(-r_1^2 + 3r_1 r_{12} + r_{12}^2 - 3r_1 r_2 + 3r_{12} r_2 - r_2^2) + \frac{1}{12}(r_1^2 - r_{12}^2 + r_2^2)\ln S. \tag{5.3}$$

Now let us devote ourselves to an evaluation of the Poisson integral with the integrand ${}_8\Lambda^{\tau i}_N$. For this integrand, ${}_8\Lambda^{\tau i}_N$, it is difficult to find all the required superpotentials. We proceed as follows. First, we split the integrand ${}_8\Lambda^{\tau i}_N$ into two groups, one whose members depend on the negative power or logarithms of S or both, and the other whose members do not.

The S-dependent integrands are

$$\mathrm{S-dependent\ parts\ of\ } \{-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l} + 2{}_4h^{i(k,l)}{}_4h^{\tau}{}_{k,l} + {}_4h^{ik}{}_{,\tau 4}h^{\tau\tau}{}_{,k} + 2{}_4h^{\tau\tau}{}_{,k6}h^{\tau[k,i]}\}.$$

For the remaining integrands in ${}_8\Lambda^{\tau i}_N$, we found all the required superpotentials except for essentially two Poisson equations whose source terms are $1/r_1^2/r_2$ and $(\ln r_1)/r_2^3$. Then in the following, we split the integrands into three groups:

(a) Direct-integration part = S-dependent group:
the integrands which depend on inverse powers of S (S^k, k : negative integer), or have logarithms of S ($\ln S$), or both.

(b) Superpotential part:
the integrands which do *not* depend on inverse powers of S, nor have logarithms of S, and for which the required superpotentials are available,

(c) Superpotential-in-series part:
the integrands which do *not* depend on inverse powers of S, nor have logarithms of S, and for which the required superpotentials are unavailable.

We mention here that splitting the integrands into the S-dependent and the S-independent group is a rather rough technique; there may be some terms in the S-dependent group for which we could find superpotentials. We have made such a classification since for the S-dependent group it seemed difficult to find superpotentials.

For example, let us take $-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}$. Then

$$\begin{aligned}
&\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\mathrm{DIP}} \equiv \mathrm{direct-integration\ part\ of\ } \{-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\} \\
&= \frac{8m_1^2 m_2}{r_1^2 S} v_1^i \left(-\frac{2}{r_1 r_{12}} + \frac{2}{r_1 r_2} + \frac{1}{r_{12} r_2} - \frac{r_{12}}{r_1^2 r_2} + \frac{r_2}{r_1^2 r_{12}}\right) \\
&\quad + \frac{16 m_1^2 m_2}{r_1 S^2} n_1^i \left(-\frac{1}{r_1 r_2}(\vec{n}_1\cdot\vec{v}_1) + \left(-\frac{1}{r_{12}^2} - \frac{2}{r_1 r_{12}} - \frac{2}{r_1^2} - \frac{r_{12}}{r_1^2 r_2} + \frac{2r_2}{r_1^2 r_{12}} + \frac{r_2^2}{r_1^2 r_{12}^2}\right)(\vec{n}_{12}\cdot\vec{v}_1)\right. \\
&\quad \left. + \left(\frac{r_2}{r_1^2 r_{12}} - \frac{1}{r_1^2} - \frac{1}{r_1 r_{12}}\right)(\vec{n}_2\cdot\vec{v}_1)\right) \\
&\quad + \frac{16 m_1^2 m_2}{S^2} n_2^i \left(-\frac{1}{r_1^3} + \frac{1}{r_{12}^3} + \frac{2}{r_1 r_{12}^2} + \frac{1}{r_1^2 r_{12}} - \frac{2r_2}{r_1^2 r_{12}^2} - \frac{r_2}{r_1^3 r_{12}} - \frac{r_2^2}{r_1^2 r_{12}^3}\right)(\vec{n}_1\cdot\vec{v}_1) \\
&\quad + \frac{16 m_1^2 m_2}{S^2} n_2^i \left(-\frac{2}{r_1^3} - \frac{1}{r_1^2 r_2} - \frac{r_{12}}{r_1^3 r_2} - \frac{r_2}{r_1 r_{12}^3} - \frac{2r_2}{r_1^2 r_{12}^2} + \frac{2r_2^2}{r_1^3 r_{12}^2} + \frac{r_2^3}{r_1^3 r_{12}^3}\right)(\vec{n}_2\cdot\vec{v}_1) \\
&\quad + (1 \leftrightarrow 2),
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\mathrm{SP}} \equiv \mathrm{superpotential\ part\ of\ } \{-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\} \\
&= \frac{4m_1^3}{r_1^5}\left(v_1^i - 3(\vec{n}_1\cdot\vec{v}_1)n_1^i\right) - \frac{16 m_1^2 v_1^2}{r_1^4}(\vec{n}_1\cdot\vec{v}_1)n_1^i \\
&\quad + \frac{2m_1^2 m_2}{r_1^2}\left(\frac{4r_{12}^2}{r_1^2 r_2^3}(\vec{n}_1\cdot\vec{v}_2)n_1^i + \left(\frac{1}{r_2^2} - \frac{r_{12}^2}{r_1^2 r_2^2} + \frac{1}{r_1^2}\right)\frac{v_2^i}{r_2}\right) - \frac{16 m_1 m_2 (\vec{n}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)}{r_1^2 r_2^2} n_1^i \\
&\quad + (1 \leftrightarrow 2),
\end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SSP} \equiv \text{superpotential} - \text{in} - \text{series part of } \{-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\} \\
&= \frac{4m_1^2m_2}{r_1^2}\left(-\frac{2}{r_2^3}(\vec{n}_1\cdot\vec{v}_2)n_1^i - \frac{2}{r_1^2r_2}(\vec{n}_1\cdot\vec{v}_2)n_1^i + \frac{1}{r_1r_2^2}(\vec{n}_1\cdot\vec{v}_2)n_2^i\right) \\
&\quad +(1\leftrightarrow 2).
\end{aligned}
\tag{5.6}
$$

Below, we first consider the superpotential part of ${}_8\Lambda^{\tau i}_N$.

A. Superpotential part

The integrands ${}_8\Lambda^{\tau i}_N-$(S $-$ dependent parts of $\{-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}+2{}_4h^{i(k,l)}{}_4h^{\tau}{}_{k,l}+{}_4h^{ik}{}_{,\tau 4}h^{\tau\tau}{}_{,k}+2{}_4h^{\tau\tau}{}_{,k6}h^{\tau[k,i]}\}$) can be simplified into a form which is independent of S. For the so-obtained S-independent group, we have to find particular solutions of Poisson equations whose source terms are,[3]

$$
\begin{aligned}
&\left\{\frac{1}{r_1^5},\frac{1}{r_1^4},\frac{1}{r_1^2},\frac{r_1^ir_1^j}{r_1^7},r_1^ir_1^j,\frac{r_1^ir_1^j}{r_1^6},\frac{r_1^ir_1^j}{r_1^4},\frac{r_1^ir_1^j}{r_1^2},\frac{r_1^ir_1^jr_1^kr_1^l}{r_1^8},\right. \\
&\frac{1}{r_1^5r_2^4},\frac{1}{r_1^5r_2^2},\frac{1}{r_1^5r_2},\frac{r_2}{r_1^5},\frac{r_2^3}{r_1^5},\frac{1}{r_1^4r_2^3},\frac{1}{r_1^4r_2},\frac{r_2^2}{r_1^4},\frac{1}{r_1^3r_2^3},\frac{1}{r_1^3r_2^2},\frac{1}{r_1^3r_2},\frac{r_2}{r_1^3},\frac{1}{r_1r_2}, \\
&\frac{r_1^ir_1^j}{r_1^6r_2^3},\frac{r_1^ir_1^j}{r_1^6r_2},\frac{r_1^ir_1^jr_2^2}{r_1^6},\frac{r_1^ir_1^j}{r_1^5r_2^3},\frac{r_1^ir_1^j}{r_1^5r_2},\frac{r_1^ir_1^j}{r_1^4r_2^3},\frac{r_1^ir_1^jr_2^2}{r_1^4},\frac{r_1^ir_1^j}{r_1^3r_2^3},\frac{r_1^ir_1^j}{r_1^3r_2},\frac{r_1^ir_1^jr_2}{r_1^3},\frac{r_1^ir_1^j}{r_2^2},\frac{r_1^ir_1^j}{r_1r_2}, \\
&\left.\frac{r_1^ir_2^j}{r_1^4r_2^3},\frac{r_1^ir_2^j}{r_1^4},\frac{r_1^ir_2^jr_2^2}{r_1^4},\frac{r_1^ir_2^j}{r_1^3r_2^3},\frac{r_1^ir_2^j}{r_1^3r_2},\frac{r_1^ir_2^jr_2}{r_1^3},\frac{r_1^ir_2^j}{r_1^2},\frac{r_1^ir_2^j}{r_1r_2},\frac{r_1^ir_1^jr_1^kr_2^l}{r_1^5r_2^3}\right\}.
\end{aligned}
\tag{5.7}
$$

It should be understood that there are Poisson equations with the same sources but with $(1\leftrightarrow 2)$ to be solved.

Our method to derive the superpotentials is heuristic; there are few guidelines available to find the required superpotentials. We proceed as follows. First, we convert all the tensorial sources into scalars with spatial derivatives. For example,

$$
\frac{r_1^ir_1^jr_1^kr_2^l}{r_1^5r_2^3} = -\frac{1}{3}\partial_{z_1^i}\partial_{z_1^j}\partial_{z_1^k}\partial_{z_2^l}\left(\frac{r_1}{r_2}\right)+\frac{1}{3}(\delta^{ij}\partial_{z_1^k}+\delta^{ik}\partial_{z_1^j}+\delta^{jk}\partial_{z_1^i})\partial_{z_2^l}\left(\frac{1}{r_1r_2}\right).
\tag{5.8}
$$

(Here and henceforth, it should be understood that in general, "scalars" can have tensorial indexes carried by $\vec{v}_A$ and $\vec{r}_{12}$, but do not have those by $\vec{r}_A$.)

Second, we find the particular solutions for Poisson equations with the scalars as sources using a formula $\Delta(f(\vec{x})g(\vec{x})) = g(\vec{x})\Delta f(\vec{x})+2\vec{\nabla}f(\vec{x})\cdot\vec{\nabla}g(\vec{x})+f(\vec{x})\Delta g(\vec{x})$ valid in N/B. We also use *superpotential chains* such as

$$
\begin{array}{ccc}
f^{(-3,-2)} & \xrightarrow{\Delta_{11}} & 6f^{(-5,-2)} \\
\downarrow \Delta_{22} & & \downarrow \Delta_{22} \\
2f^{(-3,-4)} & \xrightarrow{\Delta_{11}} & 12f^{(-5,-4)},
\end{array}
$$

where

$$
f^{(-3,-2)} = \frac{1}{r_1r_{12}^2}\ln\left(\frac{r_2}{r_1}\right),
$$

and $f^{(m,n)}$ satisfies $\Delta f^{(m,n)} = r_1^mr_2^n$. For example, for Eq. (5.8), it is easy to find a particular solution,

$$
\frac{r_1^ir_1^jr_1^kr_2^l}{r_1^5r_2^3} = \Delta\left[-\frac{1}{3}\frac{\partial^4}{\partial z_1^i\partial z_1^j\partial z_1^k\partial z_2^l}f^{(1,-1)}+\frac{1}{3}(\delta^{ij}\partial_{z_1^k}+\delta^{ik}\partial_{z_1^j}+\delta^{jk}\partial_{z_1^i})\partial_{z_2^l}\ln S\right],
$$

---

[3] What particular solutions are required depends on how we simplify the integrands.

where $f^{(1,-1)}$ is given in [33]. Another example is

$$\frac{1}{r_1^5 r_2} = \Delta\Delta_{11}\Delta_{22}\frac{1}{12}f^{(-3,1)}.$$

Following the method described above, we could find all the required particular solutions other than

$$\left\{\frac{r_1^i r_1^j}{r_1^6 r_2}, \frac{r_1^i r_1^j}{r_1^4 r_2^3}, \frac{r_1^i r_2^j}{r_1^4 r_2^3}\right\}.$$

In Appendix G, we list some of the particular solutions that we extensively used to derive the 3PN gravitational field. Other useful superpotentials are given in [28,33,40]. The necessary particular solutions can be obtained by taking derivatives of these superpotentials with respect to x^i or z_1^i or z_2^i or some combinations of them.

For example, a Poisson integral of the superpotential part of $-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}$ (Eq. (5.5)) can be evaluated as follows. We find in N/B

$$\begin{aligned}
&\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SP} \\
&= \Delta\left[4m_1^3\left(\frac{1}{15r_1^3}v_1^i + \frac{1}{5}v_1^k\partial_{ik}\frac{\ln r_1}{r_1}\right) - 2m_1^2 v_1^2 v_1^k\left(\partial_{ik}\ln r_1 + \frac{\delta^{ik}}{r_1^2}\right)\right. \\
&\quad + m_1^2 m_2 r_{12}^2 v_2^k\left(\frac{\partial^2 f^{(-2,-3)}}{\partial z_1^i\partial z_1^k} + \frac{2}{r_{12}^2}\delta^{ik}\left(f^{(-2,-3)} + f^{(-4,-1)}\right)\right) \\
&\quad \left. -16m_1m_2(\vec{v}_1\cdot\vec{v}_2)v_1^k\frac{\partial^2\ln S}{\partial z_1^i z_2^k} + (1\leftrightarrow 2)\right] \\
&\equiv \Delta SP(\tau,\vec{x}).
\end{aligned} \tag{5.9}$$

Then the surface integral in Eq. (3.31) gives

$$\begin{aligned}
&\frac{1}{-4\pi}\oint_{\partial(N/B)} dS_k\left[\frac{1}{|\vec{x}-\vec{y}|}\frac{\partial}{\partial y^k}SP(\tau,\vec{y}) - SP(\tau,\vec{y})\frac{\partial}{\partial y^k}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\right] \\
&= \frac{4m_1^3}{5r_1^3}v_1^k n_1^{<ik>}\left(\frac{23}{5} - 3\ln\epsilon R_1\right) - \frac{m_1^2 m_2}{r_{12}^2 r_2}\left(v_2^i + 2(\vec{n}_{12}\cdot\vec{v}_2)n_{12}^i + 8(\vec{n}_{12}\cdot\vec{v}_2)n_{12}^i\ln\left(\frac{r_{12}}{\epsilon R_2}\right)\right) \\
&\quad + \frac{16m_1^2 v_1^2 v_1^i}{3\epsilon R_1 r_1} - \frac{8m_1^2 m_2 v_2^i}{3\epsilon R_1 r_1 r_{12}} \\
&\quad + (1\leftrightarrow 2).
\end{aligned} \tag{5.10}$$

We shall ignore the last two terms in the second to last line of the above equation, $16m_1^2v_1^2v_1^i/(3\epsilon R_1 r_1) - 8m_1^2 m_2 v_2^i/(3\epsilon R_1 r_1 r_{12})$ (and the same terms with the label of the star exchanged, hidden in $(1\leftrightarrow 2)$), because they have negative powers of R_A. We combine the above results and evaluate the Poisson integral of $\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SP}$ as

$$\begin{aligned}
&\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SP} \\
&= \frac{12m_1^3}{5r_1^3}v_1^k n_1^{<ik>}\left(\frac{1}{5} + \ln\left(\frac{r_1}{\epsilon R_1}\right)\right) + \frac{4m_1^2}{r_1^2}v_1^2\left((\vec{n}_1\cdot\vec{v}_1)n_1^i - v_1^i\right) \\
&\quad + m_1^2 m_2\left(-\frac{2}{r_1^2 r_2}(\vec{n}_1\cdot\vec{v}_2)n_1^i + \frac{2}{r_1 r_{12} r_2}(\vec{n}_{12}\cdot\vec{v}_2)n_1^i + \frac{2}{r_1 r_{12} r_2}(\vec{n}_1\cdot\vec{v}_2)n_{12}^i\right. \\
&\quad \left. + \frac{2}{r_{12}^2 r_2}(\vec{n}_{12}\cdot\vec{v}_2)n_{12}^i\left(4\ln\left(\frac{r_1}{r_2}\right) - 4\ln\left(\frac{r_{12}}{\epsilon R_2}\right) - 1\right) + v_2^i\left(\frac{1}{r_1^2 r_2} - \frac{1}{r_{12}^2 r_2} + \frac{r_2}{r_1^2 r_{12}^2}\right)\right) \\
&\quad + \frac{16m_1m_2}{S}(\vec{v}_1\cdot\vec{v}_2)\left(\frac{v_1^i}{r_{12}} + \frac{n_1^i}{S}\left((\vec{n}_{12}\cdot\vec{v}_1) + (\vec{n}_2\cdot\vec{v}_1)\right)\right. \\
&\quad \left. + n_{12}^i\left(-\frac{1}{S}(\vec{n}_{12}\cdot\vec{v}_1) - \frac{1}{r_{12}}(\vec{n}_{12}\cdot\vec{v}_1) - \frac{1}{S}(\vec{n}_2\cdot\vec{v}_1)\right)\right) + (1\leftrightarrow 2).
\end{aligned} \tag{5.11}$$

Notice that in the above equation there appear $\ln(\epsilon R_1)$ and $\ln(\epsilon R_2)$.

In general, with the help of the superpotentials and using Eq. (3.31), we can integrate the superpotential part of ${}_8\Lambda_N^{\tau i}$. The explicit result is too long to write down here. [4]

B. Superpotential-in-series part

We could not find particular solutions for the following sources;

$$\frac{r_1^i r_1^j}{r_1^6 r_2} = \frac{1}{8}\partial_{z_1^i}\partial_{z_1^j}\left(\frac{1}{r_1^2 r_2}\right) + \Delta\left[\frac{1}{4}\delta^{ij} f^{(-4,-1)}\right],$$
$$\frac{r_1^i r_1^j}{r_1^4 r_2^3} = -\frac{1}{2}\partial_{z_1^i}\partial_{z_1^j}\left(\frac{\ln r_1}{r_2^3}\right) + \Delta\left[\frac{1}{2}\delta^{ij} f^{(-2,-3)}\right],$$
$$\frac{r_1^i r_2^j}{r_1^4 r_2^3} = \frac{1}{2}\partial_{z_1^i}\partial_{z_2^j}\left(\frac{1}{r_1^2 r_2}\right).$$

For these innocent looking sources, we did not find particular solutions valid throughout N/B in closed forms. Instead we looked for those valid near the star, say the star 1; the field we need when we evaluate the evolution equation for $P_{1\Theta}^{\tau}$ and the equation of motion for the star 1 is the field around the star 1.

Now, all the integrands classified into the superpotential-in-series part at 3PN order are found to have the following form (neglecting the m_A, $\vec{v}_A$, and $\vec{r}_{12}$ dependence appearing in actual applications of the following formulas):

$$\partial_{z_A^i}\partial_{z_{A'}^j} g(\vec{x}) \equiv \partial_{z_A^i}\partial_{z_{A'}^j}\left(\frac{(\ln r_1)^p(\ln r_2)^q}{r_1^a r_2^b}\right), \tag{5.12}$$

where a and b are integers and $p = 0, 1$, $q = 0, 1$. $A, A' = 1, 2$.

Then, we take spatial derivatives out of the Poisson integral,

$$\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\partial_{z_A^i}\partial_{z_{A'}^j} g(\vec{y}) = \partial_{z_A^i}\partial_{z_{A'}^j}\int_{N/B} d^3y\frac{g(\vec{y})}{|\vec{x}-\vec{y}|} + \partial_{z_A^i}\oint_{\partial B_{A'}} dS_j\frac{g(\vec{y})}{|\vec{x}-\vec{y}|} + \oint_{\partial B_A} dS_i\frac{\partial_{z_{A'}^j} g(\vec{y})}{|\vec{x}-\vec{y}|}. \tag{5.13}$$

For the remaining volume integral, we change the integration variable $\vec{y}$ into $\vec{y}_1$, namely, $\vec{y}_2 = \vec{r}_{12} + \vec{y}_1$. We also change the integration region N/B into N_1/B, where $N_1 \equiv \{\vec{y}||\vec{y}-\vec{z}_1| \le \mathcal{R}/\epsilon\}$,

$$\begin{aligned}
&\int_{N/B} d^3y\frac{g(\vec{y})}{|\vec{x}-\vec{y}|} \\
&= \int_{N_1/B} d^3y_1\frac{g(\vec{y}_1+\vec{z}_1)}{|\vec{r}_1-\vec{y}_1|} - z_1^k\oint_{\partial N} dS_k\frac{g(\vec{y})}{|\vec{x}-\vec{y}|} - \frac{1}{2!}z_1^k z_1^l\oint_{\partial N} dS_k\partial_{y^l}\left(\frac{g(\vec{y})}{|\vec{x}-\vec{y}|}\right) \\
&\quad -\frac{1}{3!}z_1^k z_1^l z_1^m\oint_{\partial N} dS_k\partial_{y^l}\partial_{y^m}\left(\frac{g(\vec{y})}{|\vec{x}-\vec{y}|}\right) \\
&\quad -\frac{1}{4!}z_1^k z_1^l z_1^m z_1^n\oint_{\partial N} dS_k\partial_{y^l}\partial_{y^m}\partial_{y^n}\left(\frac{g(\vec{y})}{|\vec{x}-\vec{y}|}\right) + \cdots,
\end{aligned} \tag{5.14}$$

where $\partial_{y^i} = \partial/\partial y^i$. The surface integrals and terms expressed as $\cdots$ arise due to the difference between N and N_1. See, e.g., [55]. Note that $\vec{r}_A = \vec{x} - \vec{z}_A$, where $\vec{x}$ is the field point, while $\vec{y}_A = \vec{y} - \vec{z}_A$, where $\vec{y}$ is the integral variable.

For the first volume integral on the right-hand side of the above equation, we expand $|\vec{r}_1 - \vec{y}_1|$ as

$$\frac{1}{|\vec{r}_1-\vec{y}_1|} = \sum_{c=0}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^c \mathrm{P}_c\left(\frac{\vec{r}_1\cdot\vec{y}_1}{r_1 y_1}\right), \tag{5.15}$$

[4] The number of terms are $\sim 10^3$, depending on how we simplify the result.

where $r_> = \max(r_1, y_1)$, $r_< = \min(r_1, y_1)$, and $\mathrm{P}_c(x)$ is a Legendre function of order c.

Now we split the integration region into four parts according to where the radial variable y_1 is as follows: region I, $y_1 \in [\epsilon R_1, r_1]$; region II, $y_1 \in [r_1, r_{12}-\epsilon R_2]$; region III, $y_1 \in [r_{12}-\epsilon R_2, r_{12}+\epsilon R_2]$; and region IV, $y_1 \in [r_{12}+\epsilon R_2, \mathcal{R}/\epsilon]$. In the third integral region, angular integration is incomplete due to the body zone 2 (B_2), which the Poisson integration over N/B does *not* cover.

Then first for region I,

$$\begin{aligned}\int_I \frac{d^3y_1}{|\vec{r}_1-\vec{y}_1|} g(\vec{y}) &= 2\pi \sum_{c=0} \frac{1}{r_1^{c+1}} \int_{\epsilon R_1}^{r_1} dy_1 \frac{(\ln y_1)^p}{y_1^{a-c-2}} \\ &\quad \times \int_{-1}^{1} d\cos\theta \frac{\mathrm{P}_c(\cos\theta)\mathrm{P}_c(\cos\gamma)}{2^q(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta)^{b/2}} (\ln(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta))^q \\ &= 2\pi \sum_{c=0} \frac{\mathrm{P}_c(\cos\gamma)}{r_1^{c+1} r_{12}^{a+b-c-3}} \int_{\epsilon R_1/r_{12}}^{r_1/r_{12}} d\zeta (\ln\zeta + \ln r_{12})^p \zeta^{c+2-a} \\ &\quad \times \int_{-1}^{1} \frac{dt \mathrm{P}_c(t) \left(\frac{1}{2}\ln(1-2\zeta t+\zeta^2)+\ln r_{12}\right)^q}{(1-2\zeta t+\zeta^2)^{b/2}},\end{aligned} \tag{5.16}$$

where $\cos\gamma \equiv -(\vec{r}_1 \cdot \vec{r}_{12})/r_1/r_{12}$, $t \equiv \cos\theta \equiv -(\vec{y}_1 \cdot \vec{r}_{12})/y_1/r_{12}$, and $\zeta \equiv y_1/r_{12}$.

Next for region II,

$$\begin{aligned}\int_{II} \frac{d^3y_1}{|\vec{r}_1-\vec{y}_1|} g(\vec{y}) &= 2\pi \sum_{c=0} r_1^c \int_{r_1}^{r_{12}-\epsilon R_2} dy_1 \frac{(\ln y_1)^p}{y_1^{a+c-1}} \\ &\quad \times \int_{-1}^{1} d\cos\theta \frac{\mathrm{P}_c(\cos\theta)\mathrm{P}_c(\cos\gamma)}{2^q(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta)^{b/2}} (\ln(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta))^q \\ &= 2\pi \sum_{c=0} \frac{r_1^c \mathrm{P}_c(\cos\gamma)}{r_{12}^{a+b+c-2}} \int_{r_1/r_{12}}^{1-\epsilon R_2/r_{12}} \frac{d\zeta (\ln\zeta + \ln r_{12})^p}{\zeta^{a+c-1}} \\ &\quad \times \int_{-1}^{1} \frac{dt \mathrm{P}_c(t) \left(\frac{1}{2}\ln(1-2\zeta t+\zeta^2)+\ln r_{12}\right)^q}{(1-2\zeta t+\zeta^2)^{b/2}}.\end{aligned} \tag{5.17}$$

Third for region IV,

$$\begin{aligned}\int_{IV} \frac{d^3y_1}{|\vec{r}_1-\vec{y}_1|} g(\vec{y}) &= 2\pi \sum_{c=0} r_1^c \int_{r_{12}+\epsilon R_2}^{\mathcal{R}/\epsilon} dy_1 \frac{(\ln y_1)^p}{y_1^{a+c-1}} \\ &\quad \times \int_{-1}^{1} d\cos\theta \frac{\mathrm{P}_c(\cos\theta)\mathrm{P}_c(\cos\gamma)}{2^q(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta)^{b/2}} (\ln(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta))^q \\ &= 2\pi \sum_{c=0} \frac{r_1^c \mathrm{P}_c(\cos\gamma)}{r_{12}^{a+b+c-2}} \int_{1+\epsilon R_2/r_{12}}^{\mathcal{R}/(\epsilon r_{12})} \frac{d\zeta (\ln\zeta + \ln r_{12})^p}{\zeta^{a+c-1}} \\ &\quad \times \int_{-1}^{1} \frac{dt \mathrm{P}_c(t) \left(\frac{1}{2}\ln(1-2\zeta t+\zeta^2)+\ln r_{12}\right)^q}{(1-2\zeta t+\zeta^2)^{b/2}}.\end{aligned} \tag{5.18}$$

Now for region III, the angular deficit θ_0 due to the body zone 2 is determined by

$$(\epsilon R_2)^2 = y_1^2 + r_{12}^2 - 2r_{12}y_1\cos\theta_0. \tag{5.19}$$

It is convenient to redefine ζ as $\zeta \equiv y_1/r_{12} - 1$. Then θ ranges from $-1 < \cos\theta < \cos\theta_0 \equiv 1 - \alpha(\zeta)$, where $\alpha(\zeta) = (\epsilon R_2/r_{12} - \zeta)(\epsilon R_2/r_{12} + \zeta)/2/(1+\zeta)$. Thence

$$\int_{III}\frac{d^3y_1}{|\vec{r}_1-\vec{y}_1|}g(\vec{y}) = 2\pi\sum_{c=0} r_1^c\int_{r_{12}-\epsilon R_2}^{r_{12}+\epsilon R_2} dy_1\frac{(\ln y_1)^p}{y_1^{a+c-1}}$$
$$\times\int_{-1}^{\cos\theta_0} d\cos\theta\frac{\mathrm{P}_c(\cos\theta)\mathrm{P}_c(\cos\gamma)}{2^q(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta)^{b/2}}(\ln(r_{12}^2+y_1^2-2r_{12}y_1\cos\theta))^q$$
$$= 2\pi\sum_{c=0}\frac{r_1^c\mathrm{P}_c(\cos\gamma)}{r_{12}^{a+b+c-2}}\int_{-\epsilon R_2/r_{12}}^{\epsilon R_2/r_{12}}\frac{d\zeta(\ln(1+\zeta)+\ln r_{12})^p}{(1+\zeta)^{a+c-1}}$$
$$\times\int_{-1}^{1-\alpha(\zeta)}\frac{dt\mathrm{P}_c(t)\left(\frac{1}{2}\ln(2+2\zeta-2(1+\zeta)t+\zeta^2)+\ln r_{12}\right)^q}{(2+2\zeta-2(1+\zeta)t+\zeta^2)^{b/2}}. \tag{5.20}$$

Then summing up the above results, we obtain the following formula, by which we evaluate the first volume integral on the right-hand side of Eq. (5.14):

$$\int_{N_1/B} d^3y_1\frac{g(\vec{y}_1+\vec{z}_1)}{|\vec{r}_1-\vec{y}_1|}$$
$$= 2\pi\sum_{c=0}\frac{\mathrm{P}_c(\cos\gamma)}{r_1^{c+1}r_{12}^{a+b-c-3}}\int_{\epsilon R_1/r_{12}}^{r_1/r_{12}} d\zeta(\ln\zeta+\ln r_{12})^p\zeta^{c+2-a}$$
$$\times\int_{-1}^{1}\frac{dt\mathrm{P}_c(t)\left(\frac{1}{2}\ln(1-2\zeta t+\zeta^2)+\ln r_{12}\right)^q}{(1-2\zeta t+\zeta^2)^{b/2}}$$
$$+ 2\pi\sum_{c=0}\frac{r_1^c\mathrm{P}_c(\cos\gamma)}{r_{12}^{a+b+c-2}}\int_{r_1/r_{12}}^{1-\epsilon R_2/r_{12}}\frac{d\zeta(\ln\zeta+\ln r_{12})^p}{\zeta^{a+c-1}}\int_{-1}^{1}\frac{dt\mathrm{P}_c(t)\left(\frac{1}{2}\ln(1-2\zeta t+\zeta^2)+\ln r_{12}\right)^q}{(1-2\zeta t+\zeta^2)^{b/2}}$$
$$+ 2\pi\sum_{c=0}\frac{r_1^c\mathrm{P}_c(\cos\gamma)}{r_{12}^{a+b+c-2}}\int_{-\epsilon R_2/r_{12}}^{\epsilon R_2/r_{12}}\frac{d\zeta(\ln(1+\zeta)+\ln r_{12})^p}{(1+\zeta)^{a+c-1}}$$
$$\times\int_{-1}^{1-\alpha(\zeta)}\frac{dt\mathrm{P}_c(t)\left(\frac{1}{2}\ln(2+2\zeta-2(1+\zeta)t+\zeta^2)+\ln r_{12}\right)^q}{(2+2\zeta-2(1+\zeta)t+\zeta^2)^{b/2}}$$
$$+ 2\pi\sum_{c=0}\frac{r_1^c\mathrm{P}_c(\cos\gamma)}{r_{12}^{a+b+c-2}}\int_{1+\epsilon R_2/r_{12}}^{\mathcal{R}/(\epsilon r_{12})}\frac{d\zeta(\ln\zeta+\ln r_{12})^p}{\zeta^{a+c-1}}$$
$$\times\int_{-1}^{1}\frac{dt\mathrm{P}_c(t)\left(\frac{1}{2}\ln(1-2\zeta t+\zeta^2)+\ln r_{12}\right)^q}{(1-2\zeta t+\zeta^2)^{b/2}}. \tag{5.21}$$

As an example, when the source term is $r_1^ir_2^j/r_1^4/r_2^3$, we have

$$\int_{N/B}\frac{d^3y}{-4\pi|\vec{x}-\vec{y}|}\frac{r_1^ir_2^j}{r_1^4r_2^3} = \frac{1}{2}\partial_{z_1^i}\partial_{z_2^j}F_{[1,2]}^{(-2,-1)}+O(\epsilon R_A), \tag{5.22}$$

where

$$F_{[1,2]}^{(-2,-1)} = \int_{N/B}\frac{d^3y}{-4\pi|\vec{x}-\vec{y}|}\frac{1}{y_1^2y_2} = \int_{N_1/B}\frac{d^3y_1}{-4\pi|\vec{r}_1-\vec{y}_1|}\frac{1}{y_1^2y_2}+O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2\right)$$
$$= -\frac{2}{r_{12}}+\frac{1}{r_{12}}\ln\left(\frac{r_1}{r_{12}}\right)+\frac{P_1(\cos\gamma)}{3r_{12}^2}\left(-\frac{2}{3}+\ln\left(\frac{r_1}{r_{12}}\right)\right)r_1$$
$$+\frac{P_2(\cos\gamma)}{5r_{12}^3}\left(-\frac{2}{5}+\ln\left(\frac{r_1}{r_{12}}\right)\right)r_1^2+\frac{P_3(\cos\gamma)}{7r_{12}^4}\left(-\frac{2}{7}+\ln\left(\frac{r_1}{r_{12}}\right)\right)r_1^3$$
$$+\frac{P_4(\cos\gamma)}{9r_{12}^5}\left(-\frac{2}{9}+\ln\left(\frac{r_1}{r_{12}}\right)\right)r_1^4+O\left(\frac{r_1^5}{r_{12}^5}\right)+O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2,\epsilon R_A\right). \tag{5.23}$$

$F_{[1,2]}^{(-2,-1)}$ satisfies

$$\Delta F_{[1,2]}^{(-2,-1)}-\frac{1}{r_1^2r_2} = O\left(\frac{r_1^3}{r_{12}^3}\right) \quad \text{as} \quad \mathrm{r}_1 \quad \to 0. \tag{5.24}$$

Thus, in the neighborhood of $\vec{z}_1$, $F^{(-2,-1)}_{[1,2]}$ is the required solution of the Poisson equation in the sense of Eq. (5.24). In general, $F^{(m,n)}_{[A,c]}$ denotes a function which satisfies

$$\Delta F^{(m,n)}_{[A,c]} - r_1^m r_2^n = O\left(\frac{r_A^{c+1}}{r_{12}^{c+1}}\right) \quad \text{as} \ \ \mathrm{r_A} \ \ \to 0.$$

An appropriate value of the index c depends on how many times we should take derivatives of $F^{(m,n)}_{[A,c]}$ to derive an equation of motion.

Now, to illustrate our method, let us evaluate the Poisson integral of $\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SSP}$ defined by Eq. (5.6),

$$\begin{aligned}\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SSP} &= m_1^2 m_2 v_2^k\left(4\frac{\partial^2}{\partial z_1^i z_1^k}\left(\frac{\ln r_1}{r_2^3}\right) - \frac{\partial^2}{\partial z_1^i z_1^k}\left(\frac{1}{r_1^2 r_2}\right) + 2\frac{\partial^2}{\partial z_1^k z_2^i}\left(\frac{1}{r_1^2 r_2}\right)\right)\\ &\quad +4m_1^2 m_2 v_2^i \Delta\left[-f^{(-2,-3)} - \frac{1}{2}f^{(-4,-1)}\right] + (1 \leftrightarrow 2).\end{aligned} \tag{5.25}$$

Then we evaluate the Poisson integral around $\vec{z}_1$ as

$$\begin{aligned}&\int_{N/B}\frac{d^3y}{-4\pi|\vec{x}-\vec{y}|}\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l}\right]_{\rm SSP}\\ &= m_1^2 m_2 v_2^k\left(4\frac{\partial^2}{\partial z_1^i z_1^k}F^{(\ln,-3)}_{[1,2]} - \frac{\partial^2}{\partial z_1^i z_1^k}F^{(-2,-1)}_{[1,2]} + 2\frac{\partial^2}{\partial z_1^k z_2^i}F^{(-2,-1)}_{[1,2]}\right)\\ &\quad + m_1 m_2^2 v_1^k\left(4\frac{\partial^2}{\partial z_2^i z_2^k}F^{(-3,\ln)}_{[1,2]} - \frac{\partial^2}{\partial z_2^i z_2^k}F^{(-1,-2)}_{[1,2]} + 2\frac{\partial^2}{\partial z_1^i z_2^k}F^{(-1,-2)}_{[1,2]}\right)\\ &\quad + \frac{4m_1^2 m_2}{r_{12}^2}v_2^i\left(\frac{1}{r_2}\ln\left(\frac{r_2}{\epsilon R_2}\right) + \frac{1}{r_2} - \frac{r_2}{4r_1^2}\right)\\ &\quad + \frac{4m_1 m_2^2}{r_{12}^2}v_1^i\left(\frac{1}{r_1}\ln\left(\frac{r_1}{\epsilon R_1}\right) + \frac{1}{r_1} - \frac{r_1}{4r_2^2}\right)\\ &\quad + R(\vec{x}),\end{aligned} \tag{5.26}$$

where $R(\vec{x})$ is the remainder. $F^{(\ln,-3)}_{[1,2]}$ satisfies

$$\Delta F^{(\ln,-3)}_{[1,2]} - \frac{\ln r_1}{r_2^3} = O\left(\frac{r_1^3}{r_{12}^3}\right).$$

Similar equations hold for $F^{(-3,\ln)}_{[1,2]}$ and $F^{(-1,-2)}_{[1,2]}$.

We note that for the superpotential-in-series part, there is no need to add terms corresponding to the surface integral terms in Eq. (3.31). We note also that the surface integrals in Eqs. (5.13) and (5.14) do contribute to the field in the neighborhood of the star.

We could evaluate the Poisson integral of the superpotential-in-series part of ${}_8\Lambda^{\tau i}_N$ in the neighborhood of the star 1 by means of the method described in this subsection.

C. Direct-integration part

For the direct-integration part (e.g., Eq. (5.4)), we evaluate the surface integral in Eq. (4.7) directly, while we give up deriving the corresponding contributions to the field valid throughout N/B in a closed form. In this subsection, we consider only the effect of the direct-integration part of ${}_8h^{\tau i}$ on the evolution equation for $P^{\tau}_{A\Theta}$.

Let us define the "DIP" field ${}_8h^{\tau i}_{\rm DIP}$,

$${}_8h^{\tau i}_{\rm DIP} \equiv (\text{direct} - \text{integration part of } {}_8h^{\tau i}) = 4\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}{}_8\Lambda^{\tau i}_S, \tag{5.27}$$

with $16\pi {}_8\Lambda^{\tau i}_S \equiv (\mathrm{S}-\text{dependent parts of }\{-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l} + 2{}_4h^{i(k,l)}{}_4h^{\tau}{}_{k,l} + {}_4h^{ik}{}_{,\tau}{}_4h^{\tau\tau}{}_{,k} + 2{}_4h^{\tau\tau}{}_{,k6}h^{\tau[k,i]}\})$. Then in the derivation of the evolution equation for $P^{\tau}_{1\Theta}$, the direct-integration part appears as (see Eq. (F3))

$$
\begin{aligned}
\frac{dP^\tau_{1\Theta}}{d\tau} &= -\oint_{\partial B_1} \frac{dS_k}{8\pi} {}_4h^{\tau\tau}{}_{,l\,8}h^{\tau[l,k]}_{\rm DIP} + \cdots \\
&= \oint_{\partial B_1} \frac{dS_k}{2\pi} \frac{m_2 r_2^l}{r_2^3} {}_8h^{\tau[l,k]}_{\rm DIP} + \cdots .
\end{aligned} \tag{5.28}
$$

Then it is sufficient to compute the following integral:

$$
\oint_{\partial B_1} \frac{dS_k}{2\pi} \frac{m_2 r_2^l}{r_2^3} {}_8h^{\tau[l,k]}_{\rm DIP} = \oint_{\partial B_1} dS_k \frac{2}{\pi} \frac{m_2 r_2^l}{r_2^3} \left[\int_{N/B} d^3y \frac{{}_8\Lambda_S^{\tau[l,k]}}{|\vec{x}-\vec{y}|} - \oint_{\partial(N/B)} \frac{dS_{[k\,8}\Lambda_S^{l]\tau}}{|\vec{x}-\vec{y}|} \right]. \tag{5.29}
$$

Straightforward calculation shows that ${}_8\Lambda_S^{\tau i}(\vec{y}) \sim 1/y^3$ as $y \to \infty$, and thus no contribution arises from the surface integral over ∂N. On the other hand,

$$
-\oint_{\partial B} \frac{dS_{[k\,8}\Lambda_S^{l]\tau}}{|\vec{x}-\vec{y}|} = \frac{4m_1m_2}{r_{12}^3} \left(\frac{m_1}{r_1} n_{12}^{[k} v_1^{l]} - \frac{2m_1}{3r_1} n_{12}^{[k} v_2^{l]} \right) + (1 \leftrightarrow 2). \tag{5.30}
$$

Therefore, the second surface integral in the square brackets in Eq. (5.29) gives no contribution to the evolution equation for $P^\tau_{1\Theta}$. The first integral requires special treatment, which we shall explain below.

Now let us consider an integral which has a form

$$
\oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3} \operatorname*{disc}_{\epsilon R_A} \int_{N/B} d^3y_1 \frac{f(\vec{y}_1)}{|\vec{r}_1 - \vec{y}_1|}. \tag{5.31}
$$

Here $f(\vec{x})$ carries tensorial indexes in general, but we do not write them explicitly for notational simplicity. We call this type of integral the *companion star integral*. The first integral in Eq. (5.29) is a companion star integral with $f(\vec{y}_1) = (2m_2/\pi)_8\Lambda_S^{\tau[l,k]}(\vec{y}_1 + \vec{z}_1)$ ($\vec{y}_2$ must be replaced by $\vec{r}_{12} + \vec{y}_1$ in $f(\vec{y}_1)$). In the above equation, we defined $\mathrm{disc}_{\epsilon R_A}$, which means to discard all the ϵR_A-dependent terms other than logarithms of ϵR_A. Thus, for example,

$$
\operatorname*{disc}_{\epsilon R_A} \left[\frac{1}{\epsilon R_1} \ln\left(\frac{r_{12}}{\epsilon R_2}\right) + \frac{1}{r_1} \ln\left(\frac{r_{12}}{\epsilon R_2}\right) + \frac{1}{\epsilon R_1} + \frac{1}{r_1} \right] = \frac{1}{r_1} + \frac{1}{r_1} \ln\left(\frac{r_{12}}{\epsilon R_2}\right).
$$

The symbol $\mathrm{disc}_{\epsilon R_A}$ is introduced in Eq. (5.31) to clarify that we discard ϵR_A dependence in the field *before* we evaluate the surface integrals in the general form of the 3PN equation of motion.

To evaluate a companion star integral, we first exchange the order of integration,

$$
\begin{aligned}
&\oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3} \operatorname*{disc}_{\epsilon R_A} \int_{N/B} d^3y \frac{f(\vec{y}_1)}{|\vec{r}_1 - \vec{y}_1|} \\
&= \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \int_{N/B} d^3y_1 f(\vec{y}_1) \oint_{\partial B_1'} dS_k \frac{1}{|\vec{y}_1 - r_1'\vec{n}_1|} \partial_{z_2^l} \frac{1}{|\vec{r}_{12} - r_1'\vec{n}_1|},
\end{aligned} \tag{5.32}
$$

where we defined a sphere B_1' whose center is $\vec{z}_1$ and radius is r_1' which is a constant slightly larger than ϵR_1 for any (small) ϵ ($\epsilon R_1 < r_1' << r_{12}$).

We mention here that the procedure of exchanging the order of integrations here is motivated by the works of Blanchet and Faye [39–41].

The reason we introduced r_1' is as follows. Suppose that we treat an integrand for which the superpotential is available. By calculating the Poisson integral, we have a piece of field corresponding to the integrand. The piece generally depends on ϵR_A, however we reasonably discard such R_A-dependent terms (other than logarithmic dependence) as explained in Sec. III E. Using so-obtained R_A-independent field, we evaluate the surface integrals in the general form of the 3PN equation of motion by discarding the ϵR_A dependence emerging from the surface integrals, and obtain an equation of motion. Thus discarding-ϵR_A procedure must be employed at each time, when the field is derived and then when an equation of motion is derived, not in one time. Thus r_1' was introduced to distinguish two species of ϵR_A dependence and to discard ϵR_A dependence in the right order. We show here a simple example. Let us consider the following integral:

$$
\oint_{\partial B_1} dS_k \frac{r_1^k}{r_1^3} \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} \frac{1}{y_1^2}. \tag{5.33}
$$

Using $\Delta \ln r_1 = 1/r_1^2$, we can integrate the Poisson integral and obtain the "field",

$$\int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|}\frac{1}{y_1^2} = -4\pi \ln\left(\frac{r_1}{\mathcal{R}/\epsilon}\right) + 4\pi - 4\pi\frac{\epsilon R_1}{r_1} + O\left((\epsilon R_A)^2\right).$$

Since the "body zone contribution" must have an ϵR_1 dependence hidden in the "moments" as $4\pi\epsilon R_1/r_1(+O((\epsilon R_A)^2))$ (see Sec. III E), the terms $-4\pi\epsilon R_1/r_1+O((\epsilon R_A)^2)$ should be discarded before we evaluate the "equation of motion" (the surface integral in Eq. (5.33)). The surface integral gives the "equation of motion",

$$16\pi^2\left(\ln\left(\frac{\mathcal{R}/\epsilon}{\epsilon R_1}\right)+1\right). \tag{5.34}$$

On the other hand, we can derive the "equation of motion" by first evaluating the surface integral over $\partial B_1'$,

$$\begin{aligned}
\oint_{\partial B_1} dS_k\frac{r_1^k}{r_1^3}\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\frac{1}{y_1^2} &= \int_{N/B}\frac{d^3y}{y_1^2}\oint_{\partial B_1'} dS_k\frac{r_1^k}{r_1^3}\frac{1}{|\vec{r}_1-\vec{y}_1|} \\
&= 16\pi^2\left[\int_{\epsilon R_1}^{r_1'}\frac{dy}{r_1'} + \int_{r_1'}^{\mathcal{R}/\epsilon}\frac{dy}{y_1}\right] \\
&= 16\pi^2\left(\ln\left(\frac{\mathcal{R}/\epsilon}{r_1'}\right)+1-\frac{\epsilon R_1}{r_1'}\right).
\end{aligned} \tag{5.35}$$

Thus, if we take ∂B_1 as the integral region instead of $\partial B_1'$ in the first equality in Eq. (5.35), or if we take $\lim_{r_1'\to\epsilon R_1}$ without employing $\mathrm{disc}_{\epsilon R_A}$ beforehand, we will obtain an incorrect result,

$$16\pi^2\ln\left(\frac{\mathcal{R}/\epsilon}{\epsilon R_1}\right),$$

which disagrees with Eq. (5.34).

Now let us return to Eq. (5.32). To evaluate the surface integral, we expand the integrand, supposing r_1' is small,[5]

$$\begin{aligned}
&\oint_{\partial B_1'} dS_k\frac{1}{|r_1'\vec{n}_1-\vec{y}_1|}\partial_{z_2^l}\frac{1}{|\vec{r}_{12}+r_1'\vec{n}_1|} \\
&= \partial_{z_2^l}\sum_{\substack{a=0\\b=0}}(-1)^a\frac{(2a-1)!!(2b-1)!!}{a!b!}\oint d\Omega_{\mathbf{n_1}} n_1^k n_1^{M_a} n_1^{N_b} n_{12}^{<M_a>} N_1^{<N_b>}\frac{r_1'^{a+2}}{r_{12}^{a+1}}\left\{\frac{r_1'^b}{y_1^{b+1}},\frac{y_1^b}{r_1'^{b+1}}\right\} \\
&= 4\pi\partial_{z_2^l}\sum_{a=0}(-1)^a\frac{(2a-1)!!}{(2a+3)a!}\frac{1}{r_{12}^{a+1}}n_{12}^{<M_a>}N_1^{<kM_a>}\left\{\frac{r_1'^{2a+3}}{y_1^{a+2}},y_1^{a+1}\right\} \\
&\quad -4\pi\partial_{z_2^l}\sum_{a=0}(-1)^a\frac{(2a-1)!!}{(2a+3)a!}\frac{1}{r_{12}^{a+2}}n_{12}^{<kM_a>}N_1^{<M_a>}\left\{\frac{r_1'^{2a+3}}{y_1^{a+1}},r_1'^2 y_1^a\right\},
\end{aligned} \tag{5.36}$$

where $N_1^i \equiv y_1^i/y_1$, and in $\{f,g\}$ in the above equation, f denotes the result for $r_1' < y_1$ and g denotes the result for $r_1' > y_1$. In the last equality of Eq. (5.36), we used the following formula [58]:

$$\begin{aligned}
&\frac{1}{4\pi}\oint d\Omega_{\mathbf{n_1}} n_1^k n_1^{I_a} n_1^{J_b} n_{12}^{<I_a>} N_1^{<J_b>} \\
&= \frac{a!}{(2a+1)!!}N_1^{<I_{a-1}>}n_{12}^{<kI_{a-1}>}\delta_{a,b+1} + \frac{b!}{(2b+1)!!}N_1^{<kJ_{b-1}>}n_{12}^{<J_{b-1}>}\delta_{a+1,b}.
\end{aligned} \tag{5.37}$$

Substituting Eq. (5.36) into Eq. (5.32), we establish a formula

---

[5] $0!! = (-1)!! = 1$.

$$
\begin{aligned}
&\oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3} \operatorname*{disc}_{\epsilon R_A} \int_{N/B} d^3y \frac{f(\vec{y}_1)}{|\vec{r}_1 - \vec{y}_1|} \\
&= \lim_{r_1' \to \epsilon R_A} \operatorname*{disc}_{\epsilon R_A} 4\pi \sum_{a=0} (-1)^a \frac{(2a-1)!!}{(2a+3)a!} \\
&\quad \times \left[\int_{N/B'} d^3y_1 f(\vec{y}_1) \left\{ \frac{r_1'^{2a+3}}{y_1^{a+2}} N_1^{<kM_a>} \partial_{z_2^l} \left(\frac{n_{12}^{<M_a>}}{r_{12}^{a+1}} \right) - \frac{r_1'^{2a+3}}{y_1^{a+1}} N_1^{<M_a>} \partial_{z_2^l} \left(\frac{n_{12}^{<kM_a>}}{r_{12}^{a+2}} \right) \right\} \right. \\
&\quad \left. + \int_{B_1'/B_1} d^3y_1 f(\vec{y}_1) \left\{ y_1^{a+1} N_1^{<kM_a>} \partial_{z_2^l} \left(\frac{n_{12}^{<M_a>}}{r_{12}^{a+1}} \right) - r_1'^2 y_1^a N_1^{<M_a>} \partial_{z_2^l} \left(\frac{n_{12}^{<kM_a>}}{r_{12}^{a+2}} \right) \right\} \right],
\end{aligned} \tag{5.38}
$$

where $B' \equiv B_1' \cup B_2$.

Now since we expect from the discussion in Sec. III E that an equation of motion does not depend on ϵR_A and hence r_1', in the formally infinite series in Eq. (5.38), only finite terms for which the volume integral makes r_1' dependence disappear do contribute. The r_1' dependence of the above integral can be examined by the behavior of the integrand near $\partial B_1'$. Thus, to examine r_1' dependence, we expand $f(\vec{y}_1)$ in the small y_1,[6]

$$
f(\vec{y}_1) = \sum_{p=p_0} \frac{1}{y_1^p} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}), \tag{5.39}
$$

where p_0, the lowest bound in the sum, is a finite integer. Then, r_1' dependence of the four volume integrals in Eq. (5.38) can be evaluated as

$$
\operatorname*{disc}_{\epsilon R_A} \int_{r_1'} dy_1 f(\vec{y}_1) \frac{r_1'^{2a+3}}{y_1^a} = \sum_{p=p_0} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}) \left(\frac{r_1'^{a-p+4}}{a+p-1} \delta_{1-p \neq a} - r_1'^{2a+3} \delta_{1-p,a} \ln r_1' \right), \tag{5.40a}
$$

$$
\operatorname*{disc}_{\epsilon R_A} \int_{r_1'} dy_1 f(\vec{y}_1) \frac{r_1'^{2a+3}}{y_1^{a-1}} = \sum_{p=p_0} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}) \left(\frac{r_1'^{a-p+5}}{a+p-2} \delta_{2-p \neq a} - r_1'^{2a+3} \delta_{2-p,a} \ln r_1' \right), \tag{5.40b}
$$

$$
\operatorname*{disc}_{\epsilon R_A} \int_{\epsilon R_1}^{r_1'} dy_1 f(\vec{y}_1) y_1^{a+3} = \sum_{p=p_0} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}) \left(\frac{r_1'^{a-p+4}}{a-p+4} \delta_{p-4 \neq a} + \delta_{p-4,a} \ln \left(\frac{r_1'}{\epsilon R_1} \right) \right), \tag{5.40c}
$$

$$
\operatorname*{disc}_{\epsilon R_A} \int_{\epsilon R_1}^{r_1'} dy_1 f(\vec{y}_1) r_1'^2 y_1^{a+2} = \sum_{p=p_0} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}) \left(\frac{r_1'^{a-p+5}}{a-p+3} \delta_{p-3 \neq a} + r_1'^2 \delta_{p-3,a} \ln \left(\frac{r_1'}{\epsilon R_1} \right) \right), \tag{5.40d}
$$

where $\delta_{a \neq b} = 0$ for $a = b$ and 1 otherwise. Note that $a \geq 0$. Since we take $r_1' \to \epsilon R_A$ and discard ϵR_A dependence, in the above four equations the only terms which we should retain are

$$
\operatorname*{disc}_{\epsilon R_A} \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \int_{r_1'} dy_1 f(\vec{y}_1) \frac{r_1'^{2a+3}}{y_1^a} = \sum_{p=p_0} \frac{1}{2p-5} \delta_{a,p-4} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}), \tag{5.41a}
$$

$$
\operatorname*{disc}_{\epsilon R_A} \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \int_{r_1'} dy_1 f(\vec{y}_1) \frac{r_1'^{2a+3}}{y_1^{a-1}} = \sum_{p=p_0} \frac{1}{2p-7} \delta_{a,p-5} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}), \tag{5.41b}
$$

$$
\operatorname*{disc}_{\epsilon R_A} \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \int_{\epsilon R_1}^{r_1'} dy_1 f(\vec{y}_1) y_1^{a+3} = 0, \tag{5.41c}
$$

$$
\operatorname*{disc}_{\epsilon R_A} \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \int_{\epsilon R_1}^{r_1'} dy_1 f(\vec{y}_1) r_1'^2 y_1^{a+2} = -\sum_{p=p_0} \frac{1}{2} \delta_{a,p-5} \underset{p}{\hat{f}}(\vec{N}_1 \cdot \vec{n}_{12}). \tag{5.41d}
$$

Substituting Eqs. (5.41a), (5.41b), (5.41c), and (5.41d) into Eq. (5.38), we obtain

---

[6] ${}_8\Lambda_S^{\tau i}$ has no logarithmic dependence.

$$
\begin{aligned}
&\operatorname*{disc}_{\epsilon R_A} \oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3} \operatorname*{disc}_{\epsilon R_A} \int_{N/B} d^3y \frac{f(\vec{y}_1)}{|\vec{r}_1 - \vec{y}_1|} \\
&= 4\pi \sum_{a=p_0-4\geq 0} (-1)^a \frac{(2a-1)!!}{(2a+3)^2 a!} \oint d\Omega_{\mathbf{N_1}} \underset{a+4}{\hat{f}} (\vec{N}_1 \cdot \vec{n}_{12}) N_1^{<kM_a>} \partial_{z_2^l} \left(\frac{n_{12}^{<M_a>}}{r_{12}^{a+1}} \right) \\
&+ 4\pi \sum_{a=p_0-5\geq 0} (-1)^a \frac{(2a+1)!!}{2(2a+3)^2 a!} \oint d\Omega_{\mathbf{N_1}} \underset{a+5}{\hat{f}} (\vec{N}_1 \cdot \vec{n}_{12}) N_1^{<M_a>} \partial_{z_2^l} \left(\frac{n_{12}^{<kM_a>}}{r_{12}^{a+2}} \right), \qquad (5.42)
\end{aligned}
$$

where we applied $\mathrm{disc}_{\epsilon R_A}$ to the left-hand side of Eq. (5.38) to make it clear that we discard the ϵR_A-dependent terms (other than the $\ln \epsilon R_A$ dependence) when deriving an equation of motion.

To illustrate our method, let us take $\left[-{}_4h^{kl,i}{}_4h^{\tau}{}_{k,l} \right]_{\mathrm{DIP}}$ defined by Eq. (5.4) as an integrand, for instance, in Eq. (5.42),

$$
f(\vec{y}_1) = \frac{2m_2}{\pi} \partial_{[j} \left[-{}_4h^{|kl|,i]}{}_4h^{\tau}{}_{k,l} \right]_{\mathrm{DIP}},
$$

where the vertical strokes denote that indexes between the strokes are excluded from (anti)symmetrization. We expand this integrand around z_1^i as

$$
\begin{aligned}
\frac{2m_2}{\pi} \partial_{[j} \left[-{}_4h^{|kl|,i]}{}_4h^{\tau}{}_{k,l} \right]_{\mathrm{DIP}} &= \frac{32 m_1^2 m_2^2}{\pi r_1^4 r_{12}^2} \left(-n_{12}^{[i} v_1^{j]} - 2(\vec{n}_1 \cdot \vec{v}_1) n_1^{[i} n_{12}^{j]} - 2(\vec{n}_1 \cdot \vec{n}_{12}) n_1^{[i} v_1^{j]} \right) \\
&+ O\left(\left(\frac{r_{12}}{r_1} \right)^3 \right). \qquad (5.43)
\end{aligned}
$$

Thus we see that the integrand $\partial_{[j} \left[-{}_4h^{|kl|,i]}{}_4h^{\tau}{}_{k,l} \right]_{\mathrm{DIP}}$, for which $p_0 = 4$, does not contribute to the 3PN evolution equation of $P_{1\Theta}^{\tau}$ since the angular integration of the integrand in Eq. (5.42) gives zero.

Returning to the evaluation of ${}_8h^{\tau i}_{\mathrm{DIP}}$, we apply Eq. (5.42) to $f(\vec{y}_1) = (2m_2/\pi){}_8\Lambda_S^{\tau[l,k]}(\vec{y}_1 + \vec{z}_1)$. Expanding ${}_8\Lambda_S^{\tau[l,k]}(\vec{y}_1 + \vec{z}_1)$ around z_1^i, we find that $p_0 = 4$. Evaluating the angular integral in Eq. (5.42) for ${}_8\Lambda_S^{\tau[l,k]}(\vec{y}_1 + \vec{z}_1)$, however, results in zero.

VI. HARMONIC CONDITION

In this section, we check a part of the harmonic condition,

$$
{}_{\leq 8}h^{\tau\tau}{}_{,\tau} + {}_{\leq 8}h^{\tau i}{}_{,i} = 0. \qquad (6.1)
$$

As explained in the previous section, we split the nonretarded part of the N/B contribution ${}_8h^{\tau i}_{N/Bn=0}$ into three groups: ${}_8h^{\tau i}_{\mathrm{SP}}$ (the superpotential part), ${}_8h^{\tau i}_{\mathrm{SSP}}$ (the superpotential-in-series part), and ${}_8h^{\tau i}_{\mathrm{DIP}}$ (the direct-integration part). We could obtain an explicit form of only ${}_8h^{\tau i}_{\mathrm{SP}}$, the body zone contribution ${}_8h^{\tau i}_{B}$, and the second time derivative term in the retarded field ${}_8h^{\tau i}_{N/Bn=2}$ (the last term in Eq. (5.1)). Our trick is to transform the left-hand side of Eq. (6.1) into the following form:

$$
\begin{aligned}
\mathcal{H}(\tau, \vec{x}) \equiv &{}_{\leq 8}h^{\tau\tau}{}_{,\tau} + {}_{\leq 7}h^{\tau i}{}_{,i} + {}_8h^{\tau i}_{B,i} + {}_8h^{\tau i}_{\mathrm{SP},i} + {}_8h^{\tau i}_{N/Bn=2,i} \\
&+4 \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} \frac{\partial}{\partial y^i} ({}_8\Lambda^{\tau i}_{\mathrm{SSP}} + {}_8\Lambda^{\tau i}_{S}) - 4 \oint_{\partial(N/B)} \frac{dS_i}{|\vec{x}-\vec{y}|} ({}_8\Lambda^{\tau i}_{\mathrm{SSP}} + {}_8\Lambda^{\tau i}_{S}), \qquad (6.2)
\end{aligned}
$$

where ${}_8\Lambda^{\tau i}_{\mathrm{SSP}}$ is the integrand corresponding to the superpotential-in-series part. Now not surprisingly, we could evaluate the integrals explicitly using superpotentials. For example,

$$
\begin{aligned}
\frac{\partial}{\partial y^i} 16\pi {}_8\Lambda^{\tau i}_{\mathrm{SSP}} &= \frac{\partial}{\partial y^i} \left(\frac{24 m_1^2 m_2 y_1^i (\vec{y}_1 \cdot \vec{v}_2)}{y_1^6 y_2} + \cdots \right) \\
&= \Delta \left(24 m_1^2 m_2 v_2^k \left(-\frac{1}{4} \partial_{z_1^k} f^{(-2,-3)} + \frac{r_{12}^2}{8} \partial_{z_1^k} f^{(-4,-3)} - \frac{5}{8} \partial_{z_1^k} f^{(-4,-1)} \right) \right) \\
&\quad + \cdots . \qquad (6.3)
\end{aligned}
$$

In this manner, though we could not evaluate Poisson integrals explicitly and consequently the field ${}_8h^{\tau i}$ valid throughout N/B is not available in a closed form, we checked the harmonic condition in the sense that $\mathcal{H}(\tau, \vec{x}) = 0$ throughout N/B.

VII. THE 3PN MASS-ENERGY RELATION

As explained in Sec. IV, the direct-integration part does not play any role in the evaluation of the evolution equation of $P^\tau_{A\Theta}$. Thus we can take the same method as in the evaluation of the 2.5PN equation of motion. We first express ${}_{10}\Theta^{\tau\mu}_N$ explicitly by substituting the field derived previously into Eqs. (F2) and (F3). Then evaluating the surface integrals in Eq. (4.7), we obtain the evolution equation of $P^\tau_{A\Theta}$,

$$\begin{aligned}
\left(\frac{dP^\tau_{1\Theta}}{d\tau}\right)_{\le 3\mathrm{PN}} &= -\epsilon^2\frac{m_1m_2}{r_{12}^2}\left[4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{n}_{12}\cdot\vec{v}_2)\right] \\
&+\epsilon^4\frac{m_1m_2}{r_{12}^2}\left[-\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{1}{2}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2\right. \\
&\quad-2v_1^2(\vec{n}_{12}\cdot\vec{v}_1)+4(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{V})+5v_2^2(\vec{n}_{12}\cdot\vec{v}_2)-4v_2^2(\vec{n}_{12}\cdot\vec{v}_1) \\
&\quad\left.+\frac{m_1}{r_{12}}(-4(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1))+\frac{m_2}{r_{12}}(-10(\vec{n}_{12}\cdot\vec{v}_1)+11(\vec{n}_{12}\cdot\vec{v}_2))\right] \\
&+\epsilon^6\frac{m_1m_2}{r_{12}^2}\left[-\left(\frac{3}{2}v_1^4+2v_1^2v_2^2+4v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(\frac{5}{8}v_1^4+\frac{3}{2}v_1^2v_2^2+7v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
&\quad+\left(2v_1^2+4v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-\left(2v_1^2+8v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
&\quad+\left(3v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\left(\frac{3}{4}v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)^3 \\
&\quad+2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2-6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2) \\
&\quad-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4+\frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5 \\
&\quad+\frac{m_1}{r_{12}}\left(\left(-42v_1^2-\frac{117}{4}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+60(\vec{n}_{12}\cdot\vec{v}_1)^3\right. \\
&\quad+\left(\frac{137}{4}v_1^2+\frac{37}{2}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2) \\
&\quad+\frac{297}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-\frac{219}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)-151(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2) \\
&\quad\left.+109(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-23(\vec{n}_{12}\cdot\vec{v}_2)^3\right) \\
&\quad+\frac{m_2}{r_{12}}\left(-\left(13v_1^2+18v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(\frac{17}{2}v_1^2+25v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
&\quad+26(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-28(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2) \\
&\quad\left.+16(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-20(\vec{n}_{12}\cdot\vec{v}_2)^3\right) \\
&\quad+\frac{m_1^2}{r_{12}^2}\left(\frac{33}{4}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)-\frac{m_1m_2}{r_{12}^2}\left(\frac{35}{4}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{17}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right) \\
&\quad\left.+\frac{m_2^2}{r_{12}^2}\left(-12(\vec{n}_{12}\cdot\vec{v}_1)+\frac{23}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)\right],
\end{aligned}\tag{7.1}$$

where $\vec{V}\equiv\vec{v}_1-\vec{v}_2$.

Remarkably, we can integrate Eq. (7.1) functionally,

$$P^\tau_{1\Theta}=m_1\left(1+\epsilon^2{}_2\Gamma_1+\epsilon^4{}_4\Gamma_1+\epsilon^6{}_6\Gamma_1\right)+O(\epsilon^7),\tag{7.2}$$

with

$${}_2\Gamma_1=\frac{1}{2}v_1^2+\frac{3m_2}{r_{12}},\tag{7.3}$$

$${}_4\Gamma_1=-\frac{3m_2}{2r_{12}}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{2m_2}{r_{12}}v_2^2+\frac{7m_2}{2r_{12}}v_1^2-\frac{4m_2}{r_{12}}(\vec{v}_1\cdot\vec{v}_2)+\frac{3}{8}v_1^4+\frac{7m_2^2}{2r_{12}^2}-\frac{5m_1m_2}{2r_{12}^2},\tag{7.4}$$

$$
\begin{aligned}
{}_6\Gamma_1 = {} & \frac{m_1^2 m_2}{2r_{12}^3} + \frac{21 m_1 m_2^2}{4 r_{12}^3} + \frac{5 m_2^3}{2 r_{12}^3} + \frac{5}{16} v_1^6 \\
& + \frac{m_2^2}{r_{12}^2}\left(\frac{45}{4}v_1^2 + \frac{19}{2}v_2^2 + \frac{1}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2 - 19(\vec{v}_1\cdot\vec{v}_2) - (\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) - 3(\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
& + \frac{m_1 m_2}{r_{12}^2}\left(\frac{43}{8}v_1^2 + \frac{53}{8}v_2^2 - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{53}{4}(\vec{v}_1\cdot\vec{v}_2) + \frac{85}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
& \left. - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2\right) + \frac{m_2}{r_{12}}\left(\frac{33}{8}v_1^4 + 3v_1^2v_2^2 + 2v_2^4 - 6v_1^2(\vec{v}_1\cdot\vec{v}_2) - 4v_2^2(\vec{v}_1\cdot\vec{v}_2)\right. \\
& - \frac{7}{4}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{5}{2}v_2^2(\vec{n}_{12}\cdot\vec{v}_2)^2 + 2(\vec{v}_1\cdot\vec{v}_2)^2 \\
& \left. + 2(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) + \frac{9}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4\right). \qquad (7.5)
\end{aligned}
$$

The mass-energy relation for the χ part up to 3PN order is given in Appendix E. Equations (7.2) and (E2) give the 3PN order mass-energy relation.

1. Meaning of $P^\tau_{A\Theta}$

In this section, we suggest an interesting interpretation of the mass-energy relation. First of all, we expand in an ϵ series a four-velocity of the star A normalized as $g_{\mu\nu}u^\mu_A u^\nu_A = -\epsilon^{-2}$, where $u^i_A = u^\tau_A v^i_A$. We have

$$
\begin{aligned}
u^\tau_A = {} & 1 + \epsilon^2\left[\frac{1}{2}v_A^2 + \frac{1}{4}{}_4h^{\tau\tau}\right] \\
& + \epsilon^4\left[\frac{1}{4}{}_6h^{\tau\tau} + \frac{1}{4}{}_4h^k{}_k - \frac{3}{32}({}_4h^{\tau\tau})^2 + \frac{5}{8}{}_4h^{\tau\tau}v_A^2 - {}_4h^\tau{}_k v^k_A + \frac{3}{8}v_A^4\right] \\
& + \epsilon^5\frac{1}{4}\left[{}_7h^{\tau\tau} + {}_5h^k{}_k\right] \\
& + \epsilon^6\left[\frac{1}{4}{}_8h^{\tau\tau} + \frac{1}{4}{}_6h^k{}_k + \frac{1}{16}{}_4h^{\tau\tau}{}_4h^k{}_k - \frac{3}{16}{}_4h^{\tau\tau}{}_6h^{\tau\tau} + \frac{7}{128}({}_4h^{\tau\tau})^3 + \frac{1}{4}{}_4h^{\tau k}{}_4h^\tau{}_k\right. \\
& \quad - {}_6h^\tau{}_k v^k_A - \frac{1}{4}{}_4h^{\tau\tau}{}_4h^\tau{}_k v^k_A + \frac{1}{2}{}_4h_{kl}v^k_A v^l_A + \frac{1}{8}{}_4h^k{}_k v_A^2 + \frac{5}{64}({}_4h^{\tau\tau})^2 v_A^2 \\
& \quad \left. + \frac{5}{8}{}_6h^{\tau\tau}v_A^2 - \frac{3}{2}{}_4h^\tau{}_k v^k_A v_A^2 + \frac{27}{32}{}_4h^{\tau\tau}v_A^4 + \frac{5}{16}v_A^6\right] + O(\epsilon^7). \qquad (7.6)
\end{aligned}
$$

This is a formal series since the field should be evaluated somehow at $\vec{z}_A$ while the metric derived via the point-particle description diverges at $\vec{z}_A$.

Now let us regularize this equation with Hadamard's partie finie regularization (see, e.g., [41] in the literature of the post-Newtonian approximation). Evaluating with this procedure Eq. (7.6) and $\sqrt{-g}$ expanded in ϵ up to $O(\epsilon^6)$, then comparing the result with Eq. (7.2), we find at least up to 3PN order

$$
P^\tau_{A\Theta} = m_A[\sqrt{-g}u^\tau_A]^{\rm ext}_A. \qquad (7.7)
$$

In the above equation, $[f]^{\rm ext}_A$ means that we regularize the quantity f at the star A by Hadamard's partie finie or whatever regularizations which give the same result.

We emphasize that we have never assumed this "natural" relation in advance. This relation Eq. (7.2) has been derived by solving the evolution equation for $P^\tau_{A\Theta}$ functionally.

VIII. THE 3PN MOMENTUM-VELOCITY RELATION

We now derive the 3PN momentum-velocity relation by calculating the Q^i_A integral at 3PN order. From the definition of the Q^i_A integral, Eq. (3.24),

$$
Q^i_A = \epsilon^6\oint_{\partial B_A} dS_k\left({}_{10}\Lambda^{\tau k}_N - v^k_A{}_{10}\Lambda^{\tau\tau}_N\right)y^i_A + O(\epsilon^7),
$$

we find that the calculation required is almost the same as that in the equation for $dP^\tau_{A\Theta}/d\tau$. Namely, ${}_8h^{\tau i}_{\rm DIP}$ does not contribute to the Q^i_A integral due to (i) the antisymmetry of the direct-integration part (see Eq. (5.28)), (ii) the behavior of the direct-integration part at infinity (see Eq. (5.29)), and (iii) its behavior around $\vec{z}_A$ (see Eq. (5.42). Note that ${}_8\Lambda^{\tau k}_S(\tau,\vec{z}_A+\vec{y}_A)y^i_A \sim 1/y^3_A$ in the neighborhood of the star A). Therefore, we need to compute the surface integrals in the definition of Q^i_A using the field up to 2PN order, ${}_8h^{\tau i}_B$, ${}_8h^{\tau i}_{N/Bn=2}$, ${}_8h^{\tau i}_{\rm SP}$, and ${}_8h^{\tau i}_{\rm SSP}$. We show only $Q^i_{A\Theta}$ here,

$$ {}_{\le 6}Q^i_{1\Theta} = -\epsilon^6\frac{m_1^3m_2n_{12}^{<ij>}v^j_{12}}{2r^3_{12}} = \epsilon^6\frac{d}{d\tau}\left(\frac{m_1^3m_2}{6r^3_{12}}r^i_{12}\right) = -\epsilon^6\frac{d}{d\tau}\left(\frac{1}{6}m_1^3a^i_1\right), \tag{8.1} $$

where it should be understood that a^i_A in the last expression is evaluated with the Newtonian acceleration. We show $Q^i_{A\chi}$ in Appendix E.

In the field, Q^i_A of $O(\epsilon^6)$ appears at 4PN or higher order. Thus up to 3PN order, ${}_6Q^i_A$ affects the equation of motion only through the 3PN momentum-velocity relation. As stated in Sec. IV, the nontrivial momentum-velocity relation which affects the equation of motion is that of the Θ part. This in turn motivates us to define the representative point of the star A, z^i_A, by choosing the value of $D^i_{A\Theta}$. We do not take into account $D^i_{A\chi}$ in the definition of z^i_A.

Now with $Q^i_{A\Theta}$ in hand, we obtain the momentum-velocity relation,

$$ P^i_{1\Theta} = P^\tau_{1\Theta}v^i_1 - \epsilon^6\frac{d}{d\tau}\left(\frac{1}{6}m_1^3a^i_1\right) + \epsilon^2\frac{dD^i_{1\Theta}}{d\tau}. \tag{8.2} $$

This equation suggests that we choose

$$ D^i_{A\Theta}(\tau) = \epsilon^4\frac{1}{6}m_A^3a^i_A = \epsilon^4\delta^i_{A\Theta}(\tau). \tag{8.3} $$

For a while henceforth, we shall define z^i_A by this equation. We shall later give a more convenient definition of z^i_A.

Finally, we note that the nonzero dipole moment $D^i_{A\Theta}$ of order ϵ^4 affects the 3PN field and the 3PN equation of motion in essentially the same manner as the Newtonian dipole moment affects the Newtonian field and the Newtonian equation of motion. From Eqs. (3.13), (3.14), and (3.15) (or Eqs. (A8), (A9), and (A10) for more explicit expressions), we see that $\delta^i_{A\Theta}$ appears only at ${}_{10}h^{\tau\tau}$ as

$$ h^{\tau\tau}|_{\delta_{A\Theta}} = 4\epsilon^{10}\sum_{A=1,2}\frac{\delta^k_{A\Theta}r^k_A}{r^3_A} + O(\epsilon^{11}). \tag{8.4} $$

Then the corresponding acceleration becomes

$$ m_1a^i_1|_{\delta_{A\Theta}} = -\epsilon^6\frac{3m_1\delta^k_{2\Theta}}{r^3_{12}}n_{12}^{<ik>} + \epsilon^6\frac{3m_2\delta^k_{1\Theta}}{r^3_{12}}n_{12}^{<ik>} - \epsilon^6\frac{d^2\delta^i_{1\Theta}}{d\tau^2}. \tag{8.5} $$

The last term comes from the momentum-velocity relation Eq. (8.2) and compensates the $Q^i_{A\Theta}$ integral contribution also appearing through Eq. (8.2).

Note that this change of the acceleration does not affect the conservation of the binary orbital energy,

$$ m_1a^i_1|_{\delta_{A\Theta}}v^i_1 + m_2a^i_2|_{\delta_{A\Theta}}v^i_2 = \epsilon^6\frac{d}{d\tau}\sum_{A=1,2}\left[\delta^k_{A\Theta}\frac{dv^k_A}{d\tau} - v^k_A\frac{d\delta^k_{A\Theta}}{d\tau}\right]. \tag{8.6} $$

IX. THE 3PN GRAVITATIONAL FIELD: N/B INTEGRALS

To derive a 3PN equation of motion, we have to have ${}_{10}h^{\tau\tau} + {}_8h^k{}_k$ besides ${}_8h^{\tau i}$ and the 2.5PN field. The 2.5PN field is well known [28]. (See paper II for ${}_6h^{ij}$.) At 3PN order, while the body zone contribution ${}_{\le 10}h^{\tau\tau}_B + {}_{\le 8}h^k_{Bk}$ is easily found as Eqs. (A8) and (A10) with the help of the 3PN mass-energy relation, Eqs. (7.2) and (E2), it is quite difficult to evaluate the Poisson-type integrals over the N/B region. The problem is again to find superpotentials required in the derivation of the field. Thus we have not evaluated all the Poisson integrals and we have applied the method used in the evaluation of the ${}_8h^{\tau i}$ contribution to $dP^\tau_A/d\tau$.

We shall deal with the nonretarded field from the next subsection,

$$ 4\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\left({}_{10}\Lambda^{\tau\tau}_N(\tau,\vec{y}) + {}_8\Lambda^k_{Nk}(\tau,\vec{y})\right). \tag{9.1} $$

In the last subsection, we consider the second and the fourth time derivative terms in the retarded field.

A. Superpotential part

Using the 2PN field, we first write down ${}_{10}\Lambda_N^{\tau\tau} + {}_{8}\Lambda^k_{Nk}$ explicitly, and simplify it to remove its S-dependence as much as possible. Then as in Sec. V, we split the integrand into two groups: the S-independent group and the S-dependent group. For the S-independent group, we could find the superpotentials required other than those which have the following sources in the Poisson equations:

$$\begin{aligned}
&\left\{\frac{1}{r_1^6r_2^6},\frac{1}{r_1^6r_2^4},\frac{1}{r_1^6r_2^2},\frac{r_2^3}{r_1^6},\frac{1}{r_1^4r_2^4},\frac{1}{r_1^4r_2^2},\frac{r_2}{r_1^4},\frac{1}{r_1^2r_2},\frac{r_1^ir_1^j}{r_1^6r_2},\frac{r_1^ir_1^j}{r_1^5r_2^2},\frac{r_2^5r_1^ir_1^j}{r_1^5},\frac{r_1^ir_1^j}{r_1^4r_2^3},\frac{r_1^ir_1^j}{r_1^3r_2^4},\frac{r_2^3r_1^ir_1^j}{r_1^3},\frac{r_1^ir_1^j}{r_1^2r_2^5},\right.\\
&\left.\frac{r_2r_1^ir_1^j}{r_1},\frac{r_1r_1^ir_1^j}{r_2},\frac{r_1^3r_1^ir_1^j}{r_2^3},\frac{r_1^5r_1^ir_1^j}{r_2^5},\frac{r_1^ir_2^j}{r_1^6r_2},\frac{r_1^ir_2^j}{r_1^5r_2^2},\frac{r_2^5r_1^ir_2^j}{r_1^5},\frac{r_1^ir_2^j}{r_1^4r_2^3},\frac{r_2^3r_1^ir_2^j}{r_1^3},\frac{r_2r_1^ir_2^j}{r_1}\right\}.
\end{aligned}\tag{9.2}$$

It should be understood that there are Poisson equations with the same sources but with $(1 \leftrightarrow 2)$. We shall treat the integrand corresponding to the list (9.2), the superpotential-in-series part of ${}_{10}\Lambda_N^{\tau\tau} + {}_{8}\Lambda^k_{Nk}$, in the next subsection.

For the remaining integrand of ${}_{10}\Lambda_N^{\tau\tau} + {}_{8}\Lambda^k_{Nk}$ (the superpotential part of ${}_{10}\Lambda_N^{\tau\tau} + {}_{8}\Lambda^k_{Nk}$), we need to find the superpotentials whose sources are

$$\begin{aligned}
&\left\{\frac{1}{r_1^6},\frac{1}{r_1^3},\frac{1}{r_1^6r_2^3},\frac{1}{r_1^6r_2},\frac{r_2}{r_1^6},\frac{r_2^2}{r_1^6},\frac{r_2^4}{r_1^6},\frac{r_2^6}{r_1^6},\frac{1}{r_1^5r_2^5},\frac{1}{r_1^5r_2^3},\frac{r_2^2}{r_1^5},\right.\\
&\frac{r_1^ir_1^j}{r_1^7r_2^4},\frac{r_1^ir_1^j}{r_1^7r_2^2},\frac{r_1^ir_1^j}{r_1^7r_2},\frac{r_2r_1^ir_1^j}{r_1^7},\frac{r_2^3r_1^ir_1^j}{r_1^7},\frac{r_1^ir_1^j}{r_1^6r_2^5},\frac{r_1^ir_1^j}{r_1^5r_2^4},\frac{r_1^ir_1^j}{r_1^5r_2},\frac{r_2r_1^ir_1^j}{r_1^5},\frac{r_2^3r_1^ir_1^j}{r_1^5},\frac{r_1^ir_1^j}{r_1^4r_2^5},\frac{r_1^ir_1^j}{r_1^3r_2^5},\frac{r_1^ir_1^j}{r_1r_2^5},\frac{r_1^ir_1^j}{r_1r_2^3},\\
&\frac{r_1^ir_1^j}{r_2^4},\frac{r_1r_1^ir_1^j}{r_2^5},\frac{r_1r_1^ir_1^j}{r_2^3},\frac{r_1^2r_1^ir_1^j}{r_2^2},\frac{r_1^3r_1^ir_1^j}{r_2^5},\frac{r_1^ir_2^j}{r_1^6r_2^3},\frac{r_1^ir_2^j}{r_1^6},\frac{r_2^2r_1^ir_2^j}{r_1^6},\frac{r_2^4r_1^ir_2^j}{r_1^6},\frac{r_1^ir_2^j}{r_1^5r_2^4},\frac{r_1^ir_2^j}{r_1^5r_2^3},\frac{r_1^ir_2^j}{r_1^5r_2},\frac{r_2r_1^ir_2^j}{r_1^5},\\
&\frac{r_2^3r_1^ir_2^j}{r_1^5},\frac{r_1^ir_2^j}{r_1^4},\frac{r_2^2r_1^ir_2^j}{r_1^4},\frac{r_2^4r_1^ir_2^j}{r_1^4},\frac{r_1^ir_2^j}{r_1^3r_2},\frac{r_2r_1^ir_2^j}{r_1^3},\frac{r_1^ir_2^j}{r_1^2},\frac{r_2^2r_1^ir_2^j}{r_1^2},r_1^ir_2^j,\\
&\left.\frac{r_1^ir_1^jr_1^kr_1^l}{r_1^7r_2^3},\frac{r_1^ir_1^jr_1^kr_1^l}{r_1^7r_2},\frac{r_1^ir_1^jr_1^kr_1^l}{r_1^5r_2^3},\frac{r_1^ir_1^jr_2^kr_2^l}{r_1^5r_2^5},\frac{r_1^ir_1^jr_2^kr_2^l}{r_1^5r_2^3}\right\}.
\end{aligned}\tag{9.3}$$

Here we did not list the sources for which we already find the superpotentials (see the list (5.7)).

We could derive the superpotentials corresponding to the list (9.3) using the procedure described in Sec. V A. Useful particular solutions are given in [28,33,40] and in Appendix G. Those superpotentials enable us to calculate the Poisson integral with the superpotential part as the integrand. We cannot write the result down here because of its enormous length.

B. Superpotential-in-series part

The superpotentials having the sources listed in (9.2) could not be found. Thus we employed the method explained in Sec. V B. First we transform tensorial sources into scalars. For example,

$$\frac{r_1^5r_1^ir_1^j}{r_2^5} = \frac{1}{3}\frac{\partial^2}{\partial z_1^i z_1^j}\frac{r_1^5}{r_2} + \Delta\left[\frac{1}{63}\frac{\partial^2}{\partial z_1^i z_1^j}\left(-\frac{r_1^9}{21r_2^3} + \frac{9}{7}r_{12}^2 f^{(7,-5)} - \frac{3r_1^7}{7r_2} + 3r_{12}^2 f^{(5,-3)}\right) - \frac{\delta^{ij}}{7}f^{(7,-5)}\right].$$

We apply Eqs. (5.13), (5.14), and (5.21) to the sources in (9.2). The (scalar)sources derived by making the sources in the list (9.2) scalars to which we apply Eq. (5.21) are;

$$\left\{\frac{1}{r_1^6r_2^6},\frac{1}{r_1^6r_2^4},\frac{1}{r_1^6r_2^2},\frac{r_2^3}{r_1^6},\frac{1}{r_1^4r_2^4},\frac{1}{r_1^4r_2^2},\frac{r_2}{r_1^4},\frac{1}{r_1^2r_2},\frac{r_1^5}{r_2},\frac{r_1}{r_2^4},\frac{r_1^2\ln r_1}{r_2^5},\frac{1}{r_1^2r_2},\frac{\ln r_1}{r_2^3}\right\},\tag{9.4}$$

and the same functions with their labels 1 and 2 exchanged. For these sources, we could evaluate the Poisson integrals in a similar sense to Eq. (5.24), and as a result we obtain in the neighborhood of the star 1 the field corresponding to the superpotential-in-series part.

C. Direct-integration part

We now consider the DIP field contribution to an equation of motion. At 3PN order, the DIP field appears in the integrands of the surface integrals of the general form of the 3PN equation of motion as (see Eqs. (F3) and (F4))

$$
\begin{aligned}
&{}_{10}[16\pi(-g)t^{\tau i}_{LL}]_{\rm DIP} = 2{}_4h^{\tau\tau}{}_{,k}{}_8h^{\tau[k,i]}_{\rm DIP}, &(9.5)\\
&{}_{10}[16\pi(-g)t^{ij}_{LL}]_{\rm DIP}\\
&= \frac{1}{4}(\delta^i{}_k\delta^j{}_l + \delta^i{}_l\delta^j{}_k - \delta^{ij}\delta_{kl})\left\{{}_4h^{\tau\tau,k}({}_{10}h^{\tau\tau,l}_{\rm DIP} + {}_8h^m_{{\rm DIP}\,m}{}^{,l} + 4{}_8h^{\tau l}_{{\rm DIP},\tau}) + 8{}_4h^\tau{}_m{}^{,k}{}_8h^{\tau[l,m]}_{\rm DIP}\right\}\\
&+ 2{}_4h^{\tau i}{}_{,k}{}_8h^{\tau[k,j]}_{\rm DIP} + 2{}_4h^{\tau j}{}_{,k}{}_8h^{\tau[k,i]}_{\rm DIP}. &(9.6)
\end{aligned}
$$

Here we added the ${}_8h^{\tau i}_{\rm DIP}$ contribution. (Note that in Sec. V C, we evaluated the ${}_8h^{\tau i}_{\rm DIP}$ contribution to the evolution equation for $P^\tau_{A\Theta}$, but not to an equation of motion.) Then the DIP field contribution to a 3PN acceleration denoted by $a^i_{1\rm DIP}$ becomes

$$
\begin{aligned}
&m_1a^i_{1\rm DIP}\\
&= \frac{m_1}{4\pi}\oint_{\partial B_1}d\Omega\left[\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\left({}_{10}\Lambda^{\tau\tau,i}_S + {}_8\Lambda^k_{Sk}{}^{,i} + 4{}_8\Lambda^{\tau i}_{S}{}_{,\tau} + 8v^k_1{}_8\Lambda^{\tau[i,k]}_S\right)\right]\\
&+ \frac{m_2}{4\pi}(\delta^i{}_k\delta^j{}_l + \delta^i{}_l\delta^j{}_k - \delta^{ij}\delta_{kl})\oint_{\partial B_1}dS_j\frac{r^k_2}{r^3_2}\\
&\times\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\left({}_{10}\Lambda^{\tau\tau,l}_S + {}_8\Lambda^m_{S\,m}{}^{,l} + 4{}_8\Lambda^{\tau l}_{S}{}_{,\tau} + 8v^m_2{}_8\Lambda^{\tau[l,m]}_S\right)\\
&- \frac{2m_2}{\pi}V^k\oint_{\partial B_1}dS_k\frac{r^l_2}{r^3_2}\int_{N/B}\frac{d^3y{}_8\Lambda^{\tau[l,i]}_S}{|\vec{x}-\vec{y}|} + \frac{2m_2}{\pi}v^i_2\oint_{\partial B_1}dS_j\frac{r^k_2}{r^3_2}\int_{N/B}\frac{d^3y{}_8\Lambda^{\tau[k,j]}_S}{|\vec{x}-\vec{y}|}\\
&- \frac{m_1}{4\pi}\oint_{\partial B_1}d\Omega\left[\oint_{\partial(N/B)}\frac{dS^i}{|\vec{x}-\vec{y}|}({}_{10}\Lambda^{\tau\tau}_S + {}_8\Lambda^k_{Sk}) + 4\sum_{A=1,2}v^k_A\oint_{\partial B_A}\frac{dS_k{}_8\Lambda^{\tau i}_S}{|\vec{x}-\vec{y}|}\right.\\
&\left.+ 8v^k_1\oint_{\partial(N/B)}\frac{dS_{[k}{}_8\Lambda^{i]\tau}_S}{|\vec{x}-\vec{y}|}\right]\\
&- \frac{m_2}{4\pi}(\delta^i{}_k\delta^j{}_l + \delta^i{}_l\delta^j{}_k - \delta^{ij}\delta_{kl})\oint_{\partial B_1}dS_j\frac{r^k_2}{r^3_2}\\
&\times\left[\oint_{\partial(N/B)}\frac{dS_l}{|\vec{x}-\vec{y}|}({}_{10}\Lambda^{\tau\tau}_S + {}_8\Lambda^m_{S\,m}) + 4\sum_{A=1,2}v^m_A\oint_{\partial B_A}\frac{dS_m{}_8\Lambda^{\tau l}_S}{|\vec{x}-\vec{y}|} + 8v^m_2\oint_{\partial(N/B)}\frac{dS_{[m}{}_8\Lambda^{l]\tau}_S}{|\vec{x}-\vec{y}|}\right]\\
&+ \frac{2m_2}{\pi}V^k\oint_{\partial B_1}dS_k\frac{r^l_2}{r^3_2}\oint_{\partial(N/B)}\frac{dS^{[i}{}_8\Lambda^{l]\tau}_S}{|\vec{x}-\vec{y}|} - \frac{2m_2}{\pi}v^i_2\oint_{\partial B_1}dS_j\frac{r^k_2}{r^3_2}\oint_{\partial(N/B)}\frac{dS_{[j}{}_8\Lambda^{k]\tau}_S}{|\vec{x}-\vec{y}|}, \qquad (9.7)
\end{aligned}
$$

where we used a relation

$$
\frac{d}{d\tau}\int_{N/B}d^3y\frac{{}_8\Lambda^{\tau i}_S(\tau,\vec{y})}{|\vec{x}-\vec{y}|} = \int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\frac{d}{d\tau}{}_8\Lambda^{\tau i}_S(\tau,\vec{y}) - \sum_{A=1,2}v^k_A\oint_{\partial B_A}dS_k\frac{{}_8\Lambda^{\tau i}_S(\tau,\vec{y})}{|\vec{x}-\vec{y}|}.
$$

All the integrals in Eq. (9.7) other than the first four terms can be easily evaluated. We now explain how to evaluate the first four integrals.

1. Main star integral

For the first integral in Eq. (9.7), we change the integration variable $\vec{y}$ to $\vec{y}_1$ and also change the integration region N to N_1 using Eq. (5.14). The surface integrals over ∂N can be easily evaluated. Note that $\vec{y}_2$ in the integrands must be replaced by $\vec{r}_{12}+\vec{y}_1$. For the remaining volume integral, we use computationally the same method as the

one employed by Blanchet and Faye in [40,41]. Let us consider the following integral, which we call the *main star integral*:

$$\frac{1}{4\pi}\oint_{\partial B_1} d\Omega \operatorname*{disc}_{\epsilon R_A} \int_{N_1/B} d^3y_1 \frac{f(\vec{y}_1)}{4\pi|\vec{r}_1-\vec{y}_1|}. \tag{9.8}$$

(For the definition of $\mathrm{disc}_{\epsilon R_A}$, see Sec. V C.) With the notice given below Eq. (5.32) in mind, we first exchange the order of integration,

$$\begin{aligned}
&\frac{1}{4\pi}\oint_{\partial B_1} d\Omega \operatorname*{disc}_{\epsilon R_A} \int_{N_1/B} d^3y_1 \frac{f(\vec{y}_1)}{|\vec{r}_1-\vec{y}_1|} \\
&= \lim_{r_1'\to\epsilon R_1} \operatorname*{disc}_{\epsilon R_A}\left[\int_{N_1/B'} \frac{d^3y_1}{y_1} f(\vec{y}_1) + \int_{B_1'/B_1} \frac{d^3y_1}{r_1'} f(\vec{y}_1)\right],
\end{aligned} \tag{9.9}$$

where we used

$$\oint_{\partial B_1} \frac{d\Omega}{|\vec{r}_1-\vec{y}_1|} = \begin{cases} \frac{4\pi}{r_1} & \text{for } r_1 \geq y_1, \\ \frac{4\pi}{y_1} & \text{for } r_1 < y_1. \end{cases} \tag{9.10}$$

Next we make a symmetric trace-free decomposition (STF decomposition) of the integrand on the indexes of $\vec{n}_1$,

$$f(\vec{y}_1) = \sum_{l=0} g_l(\cos\theta, y_1) n_1^{<I_l>}, \tag{9.11}$$

where $\cos\theta = -\vec{n}_{12}\cdot\vec{n}_1$. Note that $g_l(\cos\theta, y_1)$ is not necessarily a scalar. In general, $g_l(\cos\theta, y_1)$ is a tensor whose indexes are carried by $\vec{v}_A$, $\vec{r}_{12}$, or some combinations of them. Then substituting back the STF-decomposed integrand into Eq. (9.9), we have

$$\begin{aligned}
&\frac{1}{4\pi}\oint_{\partial B_1} d\Omega \operatorname*{disc}_{\epsilon R_A} \int_{N_1/B} d^3y_1 \frac{f(\vec{y}_1)}{4\pi|\vec{r}_1-\vec{y}_1|} \\
&= \frac{1}{2}\sum_{l=0} n_{12}^{<I_l>} \lim_{r_1'\to\epsilon R_1} \operatorname*{disc}_{\epsilon R_A}\left[\int_{N_1/B'} dy_1 y_1 \int dt \mathrm{P}_l(-t) g_l(t, y_1)\right. \\
&\qquad\left. + \int_{\epsilon R_1}^{r_1'} \frac{dy_1 y_1^2}{r_1'} \oint_{-1}^{1} dt \mathrm{P}_l(-t) g_l(t, y_1)\right],
\end{aligned} \tag{9.12}$$

where (below, $\vec{n}$ and $\vec{N}$ are unit vectors)

$$\int \frac{d\Omega_{\mathbf{n}}}{4\pi} n^{<I_l>} f(\vec{N}\cdot\vec{n}) = N^{<I_l>} \int \frac{d\Omega_{\mathbf{n}}}{4\pi} \mathrm{P}_l(\vec{N}\cdot\vec{n}) f(\vec{N}\cdot\vec{n}) \tag{9.13}$$

was used. Notice that in Eq. (9.12), when $y_1 \in [r_{12}-\epsilon R_2, r_{12}+\epsilon R_2]$ the body zone 2 prevents the angular integration region from being complete. The angular deficit is given by Eq. (5.19).

Now let us give an example. Take $\partial_{z_1^i}(\ln S)\partial_j(1/r_1)$ as an integrand,

$$
\oint_{\partial B_1} \frac{d\Omega}{4\pi} \operatorname*{disc}_{\epsilon R_A} \int_{N_1/B} \frac{d^3 y_1}{4\pi|\vec{r}_1 - \vec{y}_1|} \partial_{z_1^i} \ln S \partial_j \frac{1}{y_1}
$$

$$
= \oint_{\partial B_1} \frac{d\Omega}{4\pi} \operatorname*{disc}_{\epsilon R_A} \int_{N_1/B} \frac{d^3 y_1}{4\pi|\vec{r}_1 - \vec{y}_1|} \left(\frac{n_1^i n_1^j}{y_1^2 S} - \frac{n_{12}^i n_1^j}{y_1^2 S} \right)
$$

$$
= \frac{1}{2} n_{12}^{<ij>} \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \left[\int_{N_1/B'} \frac{dy_1}{y_1} \int \frac{dt \mathrm{P}_2(-t)}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} + \int_{\epsilon R_1}^{r_1'} \frac{dy_1}{r_1'} \oint_{-1}^{1} \frac{dt \mathrm{P}_2(-t)}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} \right]
$$

$$
- \frac{1}{2} n_{12}^i n_{12}^j \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \left[\int_{N_1/B'} \frac{dy_1}{y_1} \int \frac{dt \mathrm{P}_1(-t)}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} + \int_{\epsilon R_1}^{r_1'} \frac{dy_1}{r_1'} \oint_{-1}^{1} \frac{dt \mathrm{P}_1(-t)}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} \right]
$$

$$
+ \frac{1}{6} \delta^{ij} \lim_{r_1' \to \epsilon R_1} \operatorname*{disc}_{\epsilon R_A} \left[\int_{N_1/B'} \frac{dy_1}{y_1} \int \frac{dt}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} + \int_{\epsilon R_1}^{r_1'} \frac{dy_1}{r_1'} \oint_{-1}^{1} \frac{dt}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} \right]. \tag{9.14}
$$

Here

$$
\int_{N_1/B'} \frac{dy_1}{y_1} \int \frac{dt \mathrm{P}_l(-t)}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}}
= \begin{cases} \frac{5}{3r_{12}} - \frac{\pi^2}{6r_{12}} + O(\epsilon R1, r_1') & \text{for } l = 2, \\ \frac{3}{2r_{12}} - \frac{\pi^2}{6r_{12}} + O(\epsilon R1, r_1') & \text{for } l = 1, \\ \frac{2}{r_{12}} - \frac{\pi^2}{6r_{12}} + \frac{1}{r_{12}} \ln\left(\frac{r_{12}}{r_1'}\right) + O(\epsilon R1, r_1') & \text{for } l = 0, \end{cases} \tag{9.15}
$$

and

$$
\int_{\epsilon R_1}^{r_1'} \frac{dy_1}{r_1'} \oint_{-1}^{1} \frac{dt \mathrm{P}_l(-t)}{r_{12} + y_1 + \sqrt{r_{12}^2 + y_1^2 - 2y_1 r_{12} t}} = \begin{cases} O(\epsilon R_1, r_1') & l = 2, \\ O(\epsilon R_1, r_1') & l = 1, \\ \frac{1}{r_{12}} + O(\epsilon R1, r_1') & l = 0. \end{cases} \tag{9.16}
$$

Thus we obtain

$$
\oint_{\partial B_1} \frac{d\Omega}{4\pi} \operatorname*{disc}_{\epsilon R_A} \int_{N_1/B} \frac{d^3 y_1}{4\pi|\vec{r}_1 - \vec{y}_1|} \partial_{z_1^i} \ln S \partial_j \frac{1}{y_1} = \frac{2\delta^{ij}}{9r_{12}} + \frac{n_{12}^i n_{12}^j}{12 r_{12}} + \frac{\delta^{ij}}{6r_{12}} \ln\left(\frac{r_{12}}{\epsilon R_1}\right). \tag{9.17}
$$

The integrand $\partial_{z_1^i}(\ln S)\partial_j(1/r_1)$ was taken in [40] as an example of Hadamard's partie finie of Poisson integrals. The results are the same. A difference in the two results here is simply in definitions of the lower bounds of the integration. Here in our case we set ϵR_A as the lower bounds.

Now we return to the evaluation of the 3PN gravitational field. Applying Eq. (9.12) to the first integral in Eq. (9.7), where

$$
f(\vec{y}_1) = \left[\frac{\partial}{\partial y^i} {}_{10}\Lambda_S^{\tau\tau}(\tau, \vec{y}) + \frac{\partial}{\partial y^i} {}_8\Lambda_{Sk}^{k}(\tau, \vec{y}) + 4\frac{d}{d\tau} {}_8\Lambda_S^{\tau i}(\tau, \vec{y}) + 8 v_1^k \frac{\partial}{\partial y^{[k}} {}_8\Lambda_S^{i]\tau}(\tau, \vec{y}) \right] \Bigg|_{\vec{y} = \vec{y}_1 + \vec{z}_1},
$$

we could evaluate the main star integral contribution to a 3PN acceleration.

2. Companion star integral

The second, the third, and the fourth terms in Eq. (9.7) have a form of a companion star integral and thus can be evaluated by the method described in Sec. V C. The integrands have no logarithmic dependence and thus admit an

expansion in the form of Eq. (5.39), where we found $p_0=5$. Then using Eq. (5.42), we obtain the companion star integral contribution,

$$
\begin{aligned}
&\frac{m_2}{4\pi}(\delta^i{}_k\delta^j{}_l+\delta^i{}_l\delta^j{}_k-\delta^{ij}\delta_{kl})\\
&\quad\times\oint_{\partial B_1}dS_j\frac{r_2^k}{r_2^3}\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\left({}_{10}\Lambda_S^{\tau\tau,l}+{}_8\Lambda_S^{m}{}_{m}{}^{,l}+4{}_8\Lambda_S^{\tau l}{}_{,\tau}+8v_2^m{}_8\Lambda_S^{\tau[l,m]}\right)\\
&-\frac{2m_2}{\pi}V^k\oint_{\partial B_1}dS_k\frac{r_2^l}{r_2^3}\int_{N/B}d^3y\frac{{}_8\Lambda_S^{\tau[l,i]}}{|\vec{x}-\vec{y}|}+\frac{2m_2}{\pi}v_2^i\oint_{\partial B_1}dS_j\frac{r_2^k}{r_2^3}\int_{N/B}d^3y\frac{{}_8\Lambda_S^{\tau[k,j]}}{|\vec{x}-\vec{y}|}\\
&=\frac{m_1^2m_2^2}{r_{12}^5}r_{12}^i\left[\frac{46v_1^2}{9}+\frac{16v_2^2}{3}-16(\vec{n}_{12}\cdot\vec{v}_1)^2-92(\vec{v}_1\cdot\vec{v}_2)\right].
\end{aligned}
\tag{9.18}
$$

D. Retarded field

At 3PN order, we have to evaluate the integral over N/B in the second retardation expansion term,

$$
2\frac{\partial^2}{\partial\tau^2}\int_{N/B}d^3y|\vec{x}-\vec{y}|\left({}_8\Lambda_N^{\tau\tau}+{}_6\Lambda_N^k{}_k\right).
$$

This integral can be evaluated via the super-superpotential $f(\vec{y})$ satisfying ${}_8\Lambda_N^{\tau\tau}+{}_6\Lambda_N^k{}_k=\Delta^2f(\vec{y})$ as

$$
\begin{aligned}
&\int_{N/B}d^3y|\vec{x}-\vec{y}|\Delta^2f(\vec{y})\\
&=-8\pi f(\vec{x})\\
&\quad+\oint_{\partial(N/B)}dS_k\left[|\vec{x}-\vec{y}|\partial_k\Delta f(\vec{y})-\frac{y^k-x^k}{|\vec{x}-\vec{y}|}\Delta f(\vec{y})+\frac{2}{|\vec{x}-\vec{y}|}\partial_kf(\vec{y})+\frac{2(y^k-x^k)}{|\vec{x}-\vec{y}|^3}f(\vec{y})\right].
\end{aligned}
$$

The superpotential of ${}_8\Lambda_N^{\tau\tau}+{}_6\Lambda_N^k{}_k$ is easily found. (In the derivation of ${}_8h^{\tau\tau}$ and ${}_6h^{ij}$ we found the superpotentials required here.) The super-superpotentials we could not find are only those with $1/(r_1^2r_2),1/(r_1r_2^2),r_1/r_2^2$, and r_2/r_1^2 as sources. The corresponding integrands are

$$
\frac{\partial^2}{\partial\tau^2}\int_{N/B}\frac{d^3y}{4\pi|\vec{x}-\vec{y}|}\left(-\frac{21m_1^2m_2}{y_1^2y_2}-\frac{15m_1^2m_2y_2}{2r_{12}^2y_1^2}-\frac{21m_1m_2^2}{y_1y_2^2}-\frac{15m_1m_2^2y_1}{2r_{12}^2y_2^2}\right).
$$

Thus we use the method explained in Sec. V B and obtain the field near the star 1, which is sufficient to derive the equation of motion.

The N/B integral appearing in the fourth retardation expansion term can be evaluated via the following super-super-superpotentials:

$$
\frac{1}{r_1^4}=\Delta\frac{1}{2r_1^2}=\Delta^2\frac{1}{2}\ln r_1=\Delta^3\frac{1}{2}f^{(\ln,0)},
$$

$$
\frac{\vec{r}_1\cdot\vec{r}_2}{r_1^3r_2^3}=\Delta\frac{1}{2r_1r_2}=\Delta^2\frac{1}{2}\ln S=\Delta^3\frac{1}{2}f^{(\ln S)},
$$

and a formula

$$
\begin{aligned}
&\int_{N/B}d^3y|\vec{x}-\vec{y}|^3\Delta^3f(\vec{y})\\
&=-96\pi f(\vec{x})+\oint_{\partial(N/B)}dS_k\left[|\vec{x}-\vec{y}|^3\partial_k\Delta^2f(\vec{y})-3|\vec{x}-\vec{y}|(y^k-x^k)\Delta^2f(\vec{y})\right.\\
&\quad\left.+12|\vec{x}-\vec{y}|\partial_k\Delta f(\vec{y})-12\frac{y^k-x^k}{|\vec{x}-\vec{y}|}\Delta f(\vec{y})+\frac{24}{|\vec{x}-\vec{y}|}\partial_kf(\vec{y})+\frac{24(y^k-x^k)}{|\vec{x}-\vec{y}|^3}f(\vec{y})\right].
\end{aligned}
$$

It is straightforward to evaluate the surface integrals.

X. THE 3PN EQUATION OF MOTION WITH LOGARITHMIC TERMS

To obtain a 3PN equation of motion, we evaluate the surface integrals in the general form of the 3PN equation of motion Eq. (4.8) using the field ${}_8h^{\tau\tau}, {}_{\leq 6}h^{\mu\nu}$, the 3PN body zone contributions, and the 3PN N/B contributions corresponding to the superpotential part and the superpotential-in-series part. We then combine the result with the contribution from the direct-integration part. For a computational check, we have used the direct-integration method (the method with which we evaluate the direct-integration part) to evaluate the contributions to the equation of motion from all of the 3PN N/B nonretarded field, ${}_8h^{\tau i}_{N/Bn=0}$ and ${}_{10}h^{\tau\tau}_{N/Bn=0} + {}_8h^{l}_{N/Bn=0\,l}$, by assuming that they had belonged completely to a direct-integration part. As expected, we obtained the same result from two methods: the direct-integration method and the direct-integration method plus the superpotential method plus the superpotential-in-series method.

In the evaluation of the body zone field $h^{\mu\nu}_B$ (shown as Eqs. (A8), (A9), and (A10)), besides the explicitly seen energy monopole terms in these fields which must be converted into mass monopole terms via the 3PN mass-energy relation, the effects of the $Q_A^{K_l i}$ and $R_A^{K_l ij}$ integrals appearing in the 3PN field are properly taken into account through Eq. (A4). ${}_6Q^i_{A\Theta}$ given in Eq. (8.1) affects a 3PN equation of motion through the 3PN momentum-velocity relation. Since we define the representative points of the stars via Eq. (8.3), we add the corresponding acceleration given by Eq. (8.5). Furthermore, our choice of the representative points of the stars makes $D^i_{A\chi}$ appear independently of $D^i_{A\Theta}$ in the field, and hence ${}_4D^i_{A\chi}$ affects the 3PN equation of motion. In summary, ${}_{\leq 4}Q_A^{K_l i}, {}_{\leq 4}R_A^{K_l ij}, {}_6Q^i_{A\Theta}, \delta^i_{A\Theta}$, and ${}_4D^i_{A\chi}$ contributions to the 3PN field can be written as

$$ {}_{10}h^{\tau\tau} + {}_8h^k{}_k = 4\sum_{A=1,2}\frac{r_A^k}{r_A^3}\left(\delta^k_{A\Theta} + {}_4D^k_{A\chi} + {}_4R_A^{kll} - \frac{1}{2}{}_4R_A^{llk}\right) + \cdots, \tag{10.1} $$

where $\cdots$ are other contributions. The effect of this field on the equation of motion is properly taken into account via Eq. (A4) with C_{QR} replaced by $C_{QR} + C_{D_\chi} + C_{\delta_{A\Theta}}$, where $C_{\delta_{A\Theta}} = 2/3$ and $C_{D_\chi} = -350/9$. (See Eq. (8.3) for $C_{\delta_{A\Theta}}$ and Eq. (E3) for C_{D_χ}.) On the other hand, ${}_6Q^i_{A\Theta}$ and $\delta^i_{A\Theta}$ affect the equation of motion through the momentum-velocity relation in Eq. (4.8),

$$ m_1\left(\frac{dv_1^i}{d\tau}\right)_{\leq 3\mathrm{PN}} = -\epsilon^6\frac{d_6Q^i_{1\Theta}}{d\tau} - \epsilon^6\frac{d^2\delta^i_{1\Theta}}{d\tau^2} + \cdots, \tag{10.2} $$

but cancel each other out, since we chose Eq. (8.3). We note that there is no need to take into account an effect of $D^i_{A\chi}$ through the momentum-velocity relation of the χ part on the equation of motion since we evaluate the general form of the 3PN equation of motion Eq. (4.8) to derive a 3PN equation of motion. Then these contributions to a 3PN acceleration can be summarized into

$$ \begin{aligned} &(\text{the contribution to } m_1a_1^i \text{ from } {}_{\leq 4}Q_A^{K_l i}, {}_{\leq 4}R_A^{K_l ij}, {}_6Q^i_{A\Theta}, \delta^i_{A\Theta}, {}_4D^i_{A\chi}) \\ &= -\epsilon^6\left(C_{QR} + C_{D_{A\chi}} + C_{\delta_{A\Theta}}\right)\frac{m_1^3m_2^2}{2r_{12}^6}r_{12}^i - \epsilon^6\left(C_{QR} + C_{D_{A\chi}} + C_{\delta_{A\Theta}}\right)\frac{m_1^2m_2^3}{2r_{12}^6}r_{12}^i \\ &= \epsilon^6\frac{118}{9}\frac{m_1^3m_2^2}{r_{12}^6}r_{12}^i + \epsilon^6\frac{118}{9}\frac{m_1^2m_2^3}{r_{12}^6}r_{12}^i. \end{aligned} \tag{10.3} $$

Collecting these contributions mentioned from the beginning of this section, we obtain a 3PN equation of motion. However, we found that logarithmic terms having the arbitrary constants ϵR_A in their arguments survive,

$$ \begin{aligned} m_1\left(\frac{dv_1^i}{d\tau}\right)^{\text{with log}} &= m_1\left(\frac{dv_1^i}{d\tau}\right)_{\leq 2.5\mathrm{PN}} \\ &+ \epsilon^6\frac{m_1^2m_2}{r_{12}^3}\left[\frac{44m_1^2}{3r_{12}^2}n_{12}^i\ln\left(\frac{r_{12}}{\epsilon R_1}\right) - \frac{44m_2^2}{3r_{12}^2}n_{12}^i\ln\left(\frac{r_{12}}{\epsilon R_2}\right)\right. \\ &\quad\left. + \frac{22m_1}{r_{12}}\left(5(\vec{n}_{12}\cdot\vec{V})^2n_{12}^i - V^2n_{12}^i - 2(\vec{n}_{12}\cdot\vec{V})V^i\right)\ln\left(\frac{r_{12}}{\epsilon R_1}\right)\right] \\ &+ \cdots + O(\epsilon^7), \end{aligned} \tag{10.4} $$

where the acceleration through 2.5PN order, $(dv_1^i/d\tau)_{\leq 2.5\mathrm{PN}}$ is the Damour and Deruelle 2.5PN acceleration. In our formalism, we have computed it in paper II. The "$\cdots$" stands for the terms that do not include any logarithms.

Since this equation contains two arbitrary constants, the body zone radii R_A, at first sight its predictability on the orbital motion of the binary seems to be reduced. In the next section, we shall show that by reasonable redefinition of the representative points of the stars, we can remove R_A from our equation of motion. There, we show the explicit form of the 3PN equation of motion we obtained.

XI. THE 3PN EQUATION OF MOTION

The following alternative choice of the representative point of the star A removes the ϵR_A dependence in Eq. (10.4):

$$D^i_{A\Theta,\mathrm{New}}(\tau) = \epsilon^4\delta^i_{A\Theta}(\tau) - \epsilon^4\frac{22}{3}m_A^3 a_A^i \ln\left(\frac{r_{12}}{\epsilon R_A}\right) \equiv \epsilon^4\delta^i_{A\Theta}(\tau) + \epsilon^4\delta^i_{A\,\mathrm{ln}}(\tau) \equiv \epsilon^4\delta^i_A(\tau). \tag{11.1}$$

Note that this redefinition of the representative points does not affect the existence of the energy conservation, as was shown by Eq. (8.6). We can examine the effect of this redefinition onto the equation of motion using Eq. (8.5) (use $\delta^i_{A\,\mathrm{ln}}$ instead of $\delta^i_{A\Theta}$). Thence we have

$$\begin{aligned}
m_1 a_1^i|_{\delta_{A\,\mathrm{ln}}} &= -\epsilon^6\frac{3m_1\delta^k_{2\,\mathrm{ln}}}{r_{12}^3}n_{12}^{\langle ik\rangle} + \epsilon^6\frac{3m_2\delta^k_{1\,\mathrm{ln}}}{r_{12}^3}n_{12}^{\langle ik\rangle} - \epsilon^6\frac{d^2\delta^i_{1\,\mathrm{ln}}}{d\tau^2}\\
&= -\frac{44m_1^4m_2}{3r_{12}^5}n_{12}^i\ln\left(\frac{r_{12}}{\epsilon R_1}\right) + \frac{44m_1^2m_2^3}{3r_{12}^5}n_{12}^i\ln\left(\frac{r_{12}}{\epsilon R_2}\right)\\
&\quad -\frac{22m_1^3m_2}{r_{12}^4}\left(5(\vec{n}_{12}\cdot\vec{V})^2n_{12}^i - V^2n_{12}^i - 2(\vec{n}_{12}\cdot\vec{V})V^i\right)\ln\left(\frac{r_{12}}{\epsilon R_1}\right)\\
&\quad +\frac{22m_1^3m_2}{3r_{12}^4}\left(\frac{m_1}{r_{12}}n_{12}^i + \frac{m_2}{r_{12}}n_{12}^i - V^2n_{12}^i\right.\\
&\quad \left.+\,8(\vec{n}_{12}\cdot\vec{V})^2n_{12}^i - 2(\vec{n}_{12}\cdot\vec{V})V^i\right).
\end{aligned}\tag{11.2}$$

Comparing the above equation with Eq. (10.4), we easily conclude that the representative point z_A^i of the star A defined by

$$D^i_{A\Theta,\mathrm{New}}(\tau) = \epsilon^{-6}\int_{B_A} d^3y(y^i - z_A^i(\tau))\Theta_N^{\tau\tau}(\tau, y^k) = \epsilon^4\delta^i_A(\tau) \tag{11.3}$$

obeys an equation of motion free from logarithms and hence free from any ambiguity up to 3PN order inclusively.

We mention here that Blanchet and Faye [40] have already noticed that in their 3PN equation of motion a suitable coordinate transformation removes (parts of) logarithmic dependences of arbitrary parameters corresponding (roughly) to our body zone radii.[7] It is well known that choosing different values of dipole moments corresponds to the coordinate transformation.

By adding $m_1a_1^i|_{\delta_{A\,\mathrm{ln}}}$ to Eq. (10.4), we obtain our 3PN equation of motion for two spherical compact stars whose representative points are defined by Eq. (11.3),

---

[7]Unlike our case, their coordinate transformation does not remove the logarithmic dependences of their free parameters completely. The remaining logarithmic dependence was used to make their equation of motion conservative.

$$
\begin{aligned}
m_1 \frac{dv_1^i}{d\tau} = &-\frac{m_1 m_2}{r_{12}^2} n_{12}^i \\
&+ \epsilon^2 \frac{m_1 m_2}{r_{12}^2} n_{12}^i \left[-v_1^2 - 2v_2^2 + 4(\vec{v}_1 \cdot \vec{v}_2) + \frac{3}{2}(\vec{n}_{12} \cdot \vec{v}_2)^2 + \frac{5m_1}{r_{12}} + \frac{4m_2}{r_{12}} \right] \\
&+ \epsilon^2 \frac{m_1 m_2}{r_{12}^2} V^i \left[4(\vec{n}_{12} \cdot \vec{v}_1) - 3(\vec{n}_{12} \cdot \vec{v}_2) \right] \\
&+ \epsilon^4 \frac{m_1 m_2}{r_{12}^2} n_{12}^i \left[-2v_2^4 + 4v_2^2(\vec{v}_1 \cdot \vec{v}_2) - 2(\vec{v}_1 \cdot \vec{v}_2)^2 + \frac{3}{2} v_1^2 (\vec{n}_{12} \cdot \vec{v}_2)^2 + \frac{9}{2} v_2^2 (\vec{n}_{12} \cdot \vec{v}_2)^2 \right. \\
&\quad - 6(\vec{v}_1 \cdot \vec{v}_2)(\vec{n}_{12} \cdot \vec{v}_2)^2 - \frac{15}{8}(\vec{n}_{12} \cdot \vec{v}_2)^4 - \frac{57}{4}\frac{m_1^2}{r_{12}^2} - 9\frac{m_2^2}{r_{12}^2} - \frac{69}{2}\frac{m_1 m_2}{r_{12}^2} \\
&\quad + \frac{m_1}{r_{12}} \left(-\frac{15}{4} v_1^2 + \frac{5}{4} v_2^2 - \frac{5}{2}(\vec{v}_1 \cdot \vec{v}_2) + \frac{39}{2}(\vec{n}_{12} \cdot \vec{v}_1)^2 - 39(\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) \right. \\
&\quad \left. + \frac{17}{2}(\vec{n}_{12} \cdot \vec{v}_2)^2 \right) \\
&\quad \left. + \frac{m_2}{r_{12}} \left(4v_2^2 - 8(\vec{v}_1 \cdot \vec{v}_2) + 2(\vec{n}_{12} \cdot \vec{v}_1)^2 - 4(\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) - 6(\vec{n}_{12} \cdot \vec{v}_2)^2 \right) \right] \\
&+ \epsilon^4 \frac{m_1 m_2}{r_{12}^2} V^i \left[\frac{m_1}{r_{12}} \left(-\frac{63}{4}(\vec{n}_{12} \cdot \vec{v}_1) + \frac{55}{4}(\vec{n}_{12} \cdot \vec{v}_2) \right) + \frac{m_2}{r_{12}} \left(-2(\vec{n}_{12} \cdot \vec{v}_1) - 2(\vec{n}_{12} \cdot \vec{v}_2) \right) \right. \\
&\quad + v_1^2 (\vec{n}_{12} \cdot \vec{v}_2) + 4v_2^2 (\vec{n}_{12} \cdot \vec{v}_1) - 5v_2^2 (\vec{n}_{12} \cdot \vec{v}_2) - 4(\vec{v}_1 \cdot \vec{v}_2)(\vec{n}_{12} \cdot \vec{V}) \\
&\quad \left. - 6(\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2)^2 + \frac{9}{2}(\vec{n}_{12} \cdot \vec{v}_2)^3 \right] \\
&+ \epsilon^5 \frac{4m_1^2 m_2}{5r_{12}^3} \left[n_{12}^i (\vec{n}_{12} \cdot \vec{V}) \left(-6\frac{m_1}{r_{12}} + \frac{52}{3}\frac{m_2}{r_{12}} + 3V^2 \right) + V^i \left(2\frac{m_1}{r_{12}} - 8\frac{m_2}{r_{12}} - V^2 \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^6\frac{m_1m_2}{r_{12}^2}n_{12}^i\left[\frac{35}{16}(\vec{n}_{12}\cdot\vec{v}_2)^6-\frac{15}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4v_1^2+\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^4(\vec{v}_1\cdot\vec{v}_2)\right.\\
&\quad+3(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)^2-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^4v_2^2+\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2v_2^2-12(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)v_2^2\\
&\quad-2(\vec{v}_1\cdot\vec{v}_2)^2v_2^2+\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^4+4(\vec{v}_1\cdot\vec{v}_2)v_2^4-2v_2^6\\
&\quad+\frac{m_1}{r_{12}}\left(-\frac{171}{8}(\vec{n}_{12}\cdot\vec{v}_1)^4+\frac{171}{2}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)\right.\\
&\quad-\frac{723}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{383}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3-\frac{455}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4\\
&\quad+\frac{229}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2-\frac{205}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2+\frac{191}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2-\frac{91}{8}v_1^4\\
&\quad-\frac{229}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)+244(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)-\frac{225}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)\\
&\quad+\frac{91}{2}v_1^2(\vec{v}_1\cdot\vec{v}_2)-\frac{177}{4}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{229}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2^2-\frac{283}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\\
&\quad\left.+\frac{259}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(\vec{v}_1\cdot\vec{v}_2)v_2^2-\frac{81}{8}v_2^4\right)\\
&\quad+\frac{m_2}{r_{12}}\left(-6(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2+12(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3\right.\\
&\quad+6(\vec{n}_{12}\cdot\vec{v}_2)^4+4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+12(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)+4(\vec{v}_1\cdot\vec{v}_2)^2\\
&\quad\left.-4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2-12(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2-8(\vec{v}_1\cdot\vec{v}_2)v_2^2+4v_2^4\right)\\
&\quad+\frac{m_2^2}{r_{12}^2}\left(-(\vec{n}_{12}\cdot\vec{v}_1)^2+2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{43}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2+18(\vec{v}_1\cdot\vec{v}_2)-9v_2^2\right)\\
&\quad+\frac{m_1m_2}{r_{12}^2}\left(\frac{415}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2-\frac{375}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{1113}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2\right.\\
&\quad\left.-\frac{615\pi^2}{64}(\vec{n}_{12}\cdot\vec{V})^2+18v_1^2+\frac{123\pi^2}{64}V^2+33(\vec{v}_1\cdot\vec{v}_2)-\frac{33}{2}v_2^2\right)\\
&\quad+\frac{m_1^2}{r_{12}^2}\left(-\frac{2069}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2+543(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)-\frac{939}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{471}{8}v_1^2\right.\\
&\quad\left.-\frac{357}{4}(\vec{v}_1\cdot\vec{v}_2)+\frac{357}{8}v_2^2\right)\\
&\quad\left.+\frac{16m_2^3}{r_{12}^3}+\frac{m_1^2m_2}{r_{12}^3}\left(\frac{547}{3}-\frac{41\pi^2}{16}\right)-\frac{13m_1^3}{12r_{12}^3}+\frac{m_1m_2^2}{r_{12}^3}\left(\frac{545}{3}-\frac{41\pi^2}{16}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^6\frac{m_1m_2}{r_{12}^2}V^i\left[\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5-\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3v_1^2\right.\\
&\quad+6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)-6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2)-2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2\\
&\quad-12(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2+12(\vec{n}_{12}\cdot\vec{v}_2)^3v_2^2+(\vec{n}_{12}\cdot\vec{v}_2)v_1^2v_2^2-4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)v_2^2\\
&\quad+8(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)v_2^2+4(\vec{n}_{12}\cdot\vec{v}_1)v_2^4-7(\vec{n}_{12}\cdot\vec{v}_2)v_2^4+\frac{m_2}{r_{12}}\left(-2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)\right.\\
&\quad+8(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2+2(\vec{n}_{12}\cdot\vec{v}_2)^3+2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)+4(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)\\
&\quad\left.-2(\vec{n}_{12}\cdot\vec{v}_1)v_2^2-4(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\right)+\frac{m_1}{r_{12}}\left(-\frac{243}{4}(\vec{n}_{12}\cdot\vec{v}_1)^3+\frac{565}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)\right.\\
&\quad-\frac{269}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{95}{12}(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{207}{8}(\vec{n}_{12}\cdot\vec{v}_1)v_1^2-\frac{137}{8}(\vec{n}_{12}\cdot\vec{v}_2)v_1^2\\
&\quad\left.-36(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)+\frac{27}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+\frac{81}{8}(\vec{n}_{12}\cdot\vec{v}_1)v_2^2+\frac{83}{8}(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\right)\\
&\quad+\frac{m_2^2}{r_{12}^2}\left(4(\vec{n}_{12}\cdot\vec{v}_1)+5(\vec{n}_{12}\cdot\vec{v}_2)\right)\\
&\quad+\frac{m_1m_2}{r_{12}^2}\left(-\frac{307}{8}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{479}{8}(\vec{n}_{12}\cdot\vec{v}_2)+\frac{123\pi^2}{32}(\vec{n}_{12}\cdot\vec{V})\right)\\
&\quad\left.+\frac{m_1^2}{r_{12}^2}\left(\frac{311}{4}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{357}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)\right]\\
&\quad+O(\epsilon^7),
\end{aligned}
\tag{11.4}
$$

in the harmonic gauge.

Now we list some features of our 3PN equation of motion. In the test-particle limit, our 3PN equation of motion coincides with a geodesic equation for a test-particle in the Schwarzschild metric in the harmonic coordinate (up to 3PN order). With the help of the formulas developed in [42], we have checked the Lorentz invariance of Eq. (11.4) (in the post-Newtonian perturbative sense). Also, we have checked that our 3PN acceleration admits a conserved energy of the binary orbital motion (modulo the 2.5PN radiation reaction effect). In fact, the energy of the binary E associated with Eq. (11.4) is

$$
\begin{aligned}
E &= \frac{1}{2}m_1v_1^2-\frac{m_1m_2}{2r_{12}}\\
&\quad+\epsilon^2\left[\frac{3}{8}m_1v_1^4+\frac{m_1^2m_2}{2r_{12}^2}+\frac{m_1m_2}{2r_{12}}\left(3v_1^2-\frac{7}{2}(\vec{v}_1\cdot\vec{v}_2)-\frac{1}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^4\left[\frac{5}{16}m_1v_1^6-\frac{m_1^3m_2}{2r_{12}^3}-\frac{19m_1^2m_2^2}{8r_{12}^3}\right.\\
&+\frac{m_1^2m_2}{2r_{12}^2}\left(-3v_1^2+\frac{7}{2}v_2^2+\frac{29}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2-\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+(\vec{n}_{12}\cdot\vec{v}_2)^2\right)\\
&+\frac{m_1m_2}{4r_{12}}\left(\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)+\frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2\right.\\
&-\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2+\frac{21}{2}v_1^4+\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)+3(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)\\
&\left.\left.-\frac{55}{2}v_1^2(\vec{v}_1\cdot\vec{v}_2)+\frac{17}{2}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{31}{4}v_1^2v_2^2\right)\right]\\
&+\epsilon^6\left[\frac{35}{128}m_1v_1^8+\frac{3m_1^4m_2}{8r_{12}^4}+\frac{469m_1^3m_2^2}{18r_{12}^4}\right.\\
&+\frac{m_1^2m_2^2}{2r_{12}^3}\left(\frac{547}{6}(\vec{n}_{12}\cdot\vec{v}_1)^2-\frac{3115}{24}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)-\frac{123\pi^2}{32}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{V})\right.\\
&\left.-\frac{575}{9}v_1^2+\frac{41\pi^2}{32}(\vec{V}\cdot\vec{v}_2)+\frac{4429}{72}(\vec{v}_1\cdot\vec{v}_2)\right)\\
&+\frac{m_1^3m_2}{2r_{12}^3}\left(-\frac{437}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2+\frac{317}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+3(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{301}{12}v_1^2\right.\\
&\left.-\frac{337}{12}(\vec{v}_1\cdot\vec{v}_2)+\frac{5}{2}v_2^2\right)\\
&+\frac{m_1m_2}{r_{12}}\left(-\frac{5}{16}(\vec{n}_{12}\cdot\vec{v}_1)^5(\vec{n}_{12}\cdot\vec{v}_2)-\frac{5}{16}(\vec{n}_{12}\cdot\vec{v}_1)^4(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{5}{32}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)^3\right.\\
&+\frac{19}{16}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)v_1^2+\frac{15}{16}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2+\frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3v_1^2\\
&+\frac{19}{16}(\vec{n}_{12}\cdot\vec{v}_2)^4v_1^2-\frac{21}{16}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^4-2(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^4+\frac{55}{16}v_1^6-\frac{19}{16}(\vec{n}_{12}\cdot\vec{v}_1)^4(\vec{v}_1\cdot\vec{v}_2)\\
&-(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)-\frac{15}{32}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)\\
&+\frac{45}{16}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2(\vec{v}_1\cdot\vec{v}_2)+\frac{5}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2(\vec{v}_1\cdot\vec{v}_2)+\frac{11}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2(\vec{v}_1\cdot\vec{v}_2)\\
&-\frac{139}{16}v_1^4(\vec{v}_1\cdot\vec{v}_2)-\frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)^2+\frac{5}{16}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2+\frac{41}{8}v_1^2(\vec{v}_1\cdot\vec{v}_2)^2\\
&\left.+\frac{1}{16}(\vec{v}_1\cdot\vec{v}_2)^3-\frac{45}{16}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2v_2^2-\frac{23}{32}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2v_2^2+\frac{79}{16}v_1^4v_2^2-\frac{161}{32}v_1^2v_2^2(\vec{v}_1\cdot\vec{v}_2)\right)\\
&+\frac{m_1^2m_2}{r_{12}^2}\left(-\frac{49}{8}(\vec{n}_{12}\cdot\vec{v}_1)^4+\frac{75}{8}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)-\frac{187}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{11}{2}v_1^4\right.\\
&+\frac{247}{24}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{49}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2+\frac{81}{8}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2-\frac{21}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2\\
&-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)-\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+\frac{21}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)-27v_1^2(\vec{v}_1\cdot\vec{v}_2)\\
&+\frac{55}{2}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{49}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2^2-\frac{27}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2+\frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2+\frac{55}{4}v_1^2v_2^2\\
&\left.\left.-28(\vec{v}_1\cdot\vec{v}_2)v_2^2+\frac{135}{16}v_2^4\right)\right]+(1\leftrightarrow 2)+O(\epsilon^7). \qquad (11.5)
\end{aligned}
$$

This orbital energy of the binary is computed based on that found in Blanchet and Faye [40], the relation between their 3PN equation of motion and our result described in Sec. XII 1 below, and Eq. (8.6). (After constructing E given as Eq. (11.5), we have checked that our 3PN equations of motion make E conserved.)

We note that Eq. (10.4) gives a correct geodesic equation in the test-particle limit, is Lorentz invariant, and admits the conserved energy. These facts can be seen by the form of $a_1^i|_{\delta_{A\,\ln}}$, Eq. (11.2); it is zero when $m_1 \to 0$, is Lorentz invariant up to 3PN order, and is the effect of the mere redefinition of the dipole moments which does not break energy conservation.

Finally, we mention here two computational details. In the course of calculation, $\vec{z}_1$, $\vec{z}_2$ appears independently, that is, not in a form as $\vec{r}_{12}$. This can be seen, for instance, from Eq. (5.14). As another example, the surface integral over the near zone boundary in Eq. (3.31) in general gives terms depending on $\vec{z}_A$ explicitly. All such $\vec{z}_A$ dependences, when collected in the equation of motion, are combined into $\vec{r}_{12}$. We have retained during our calculation $\mathcal{R}$-dependent terms with positive powers of $\mathcal{R}$ or logarithms of $\mathcal{R}$. As stated below Eq. (3.9), it is a good computational check to show that our equation of motion does not depend on $\mathcal{R}$ physically. In fact, we found that $\mathcal{R}$-dependent terms canceled each other out in the final result. There was no need to employ a gauge transformation to remove such $\mathcal{R}$ dependences.

XII. COMPARISON AND SUMMARY

1. Comparison

By comparing Eq. (11.4) with the Blanchet and Faye 3PN equation of motion [40], we find the following relationship:

$$m_1\vec{a}_1^{\text{this work}} = m_1(\vec{a}_1^{\text{BF}})_{\lambda=-\frac{1987}{3080}} + m_1\vec{a}_1|_{\delta_{A\,\ln}} + m_1\vec{a}_1|_{\delta_{A,\text{BF}}}, \tag{12.1}$$

where $m_1\vec{a}_1^{\text{this work}}$ is the 3PN acceleration given in Eq. (11.4), $(\vec{a}_1^{\text{BF}})_{\lambda=-1987/3080}$ is the Blanchet and Faye 3PN acceleration with $\lambda = -1987/3080$, and $m_1\vec{a}_1|_{\delta_{A\,\ln}}$ is given in Eq. (11.2) with ϵR_A replaced by r'_A for notational consistency with the Blanchet and Faye 3PN equation of motion shown in [40]. $m_1\vec{a}_1|_{\delta_{A,\text{BF}}}$ is an acceleration induced by the following dipole moments of the stars:

$$\delta^i_{A,\text{BF}} = -\frac{3709}{1260}m_A^3 a_A^i. \tag{12.2}$$

We can compute $m_1 a_1^i|_{\delta_{A,\text{BF}}}$ by substituting $\delta^i_{A,\text{BF}}$ instead of $\delta^i_{A\Theta}$ into Eq. (8.5). Thus, by choosing the dipole moments,

$$D^i_{A\Theta,\text{BF}} = \epsilon^4\delta^i_{A\Theta} - \epsilon^4\delta^i_{A,\text{BF}}, \tag{12.3}$$

we have the 3PN equation of motion in completely the same form as $(\vec{a}_1^{\text{BF}})_{\lambda=-1987/3080}$. In other words, our 3PN equation of motion physically agrees with $(\vec{a}_1^{\text{BF}})_{\lambda=-1987/3080}$ modulo the definition of the dipole moments (or equivalently, the coordinate transformation under the harmonic coordinate condition). In [47], we have shown some arguments that support this conclusion.

The value of λ that we found, $\lambda = -1987/3080$, is perfectly consistent with the relation (1.1) and the result of [38] ($\omega_{\text{static}} = 0$).

2. Summary

To deal with strongly self-gravitating objects such as neutron stars, we have used the surface integral approach with the strong field point-particle limit. The surface integral approach is achieved by using the local conservation of the energy momentum, which led us to the general form of the equation of motion which is expressed entirely in terms of surface integrals. The use of the strong field point-particle limit and the surface integral approach makes our 3PN equation of motion applicable to inspiraling compact binaries which consist of strongly self-gravitating regular stars (modulo the scalings imposed on the initial hypersurface). Our 3PN equation of motion depends only on masses of the stars and is independent of their internal structure such as their density profiles or radii. Thus our result supports the strong equivalence principle up to 3PN order.

The multipole moments previously defined in papers I and II are found to be unsatisfactory at 3PN order in the sense that these moments are not defined in a proper reference coordinate and consequently they contain monopole terms. By taking account of the effect of Lorentz contraction on these moments, we succeeded in extracting cleanly the monopole terms from these multipole moments up to the required order.

At 3PN order, it does not seem possible to derive the field in a closed form. This is because not all the superpotentials required are available, and thus we could not evaluate all the Poisson-type N/B integrals. Some of the integrands allow us to derive superpotentials in series forms in the neighborhood of a star. For others, we have adopted an idea that Blanchet and Faye have used in [39–41]. The idea is that while abandoning complete derivation of the 3PN gravitational field throughout N/B, one exchanges the order of integration. We first evaluate the surface integrals

in the evolution equation for the energy of a star and the general form of the equation of motion, and then we evaluate the remaining volume integrals. Using these methods, we first derived the 3PN mass-energy relation and the momentum-velocity relation. The 3PN mass-energy relation admits a natural interpretation. We then evaluated the surface integrals in the general form of the equation of motion, and obtained an equation of motion up to 3PN order of accuracy.

At 3PN order, our equation of motion contains logarithms of the body zone radii R_A. Practically, we cannot discard R_A dependences if R_A is in logarithms. We showed that we could remove the logarithms by suitable redefinition of the representative points of the stars. Thus we could transform our 3PN equation of motion into an unambiguous equation which does not contain any arbitrarily introduced free parameters.

Our so-obtained 3PN equation of motion agrees physically (modulo a definition of the representative points of the stars) with the result derived by Blanchet and Faye [40] with $\lambda = -1987/3080$, which is consistent with Eq. (1.1) and $\omega_{\rm static} = 0$ reported by Damour, Jaranowski, and Schäfer [38]. This result indirectly supports the validity of the dimensional regularization in the ADM canonical approach in the ADMTT gauge.

Blanchet and Faye [39,40] introduced four arbitrary parameters. In Hadamard's partie finie regularization, one has to introduce a sphere around each singular point (representing a point-mass) whose radius is a free parameter. In their framework, regularizations are employed in the evaluation of both a gravitational field having two singular points and two equations of motion. Since there is *a priori* no reason to expect that the spheres introduced for the evaluation of the field and the equations of motion coincide, there arise four arbitrary parameters. This is in contrast with our formalism where each body zone introduced in the evaluation of the field is inevitably the same as the body zone with which we defined the energy and the three-momentum of each star for which we derived our equation of motion.

Actually, the redefinition of the representative points in our formalism corresponds to the gauge transformation in [40], and only two of the four parameters remain in [40]. Then they have used one of the remaining two free parameters to ensure the energy conservation and there remains only one arbitrary parameter λ which they could not fix in their formalism.

On the other hand, our 3PN equation of motion has no ambiguous parameter, admits conservation of an orbital energy of the binary system (when we neglect the 2.5 PN radiation reaction effect), and respects Lorentz invariance in the post-Newtonian perturbative sense. We emphasize that we do not need to *a posteriori* adjust some parameters to make our 3PN equation of motion satisfy the above three physical features.

Finally, we note here that Blanchet *et al.* [43] have recently obtained the same value of λ by computing a 3PN equation of motion in the harmonic gauge using the dimensional regularization.

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APPENDIX A: $Q_A^{K_L I}$ AND $R_A^{K_L IJ}$

In this section we evaluate $Q_A^{K_l i}$ and $R_A^{K_l ij}$ up to $O(\epsilon^4)$, the order required to compute the 3PN field. In the following, we shall omit the ϵR_A dependence as explained in the appendix of paper II. In the evaluation of the field up to 3PN order, ${}_{\le 10}h^{\tau\tau}$ and ${}_{\le 8}h^{\mu i}$, we use ${}_{\le 8}\Lambda_N^{\mu\nu}$ as the integrand in the surface integrals of $Q_A^{K_l i}$ and $R_A^{K_l ij}$. Then straightforward calculation gives

$$Q_A^{K_l i} = \epsilon^{-4} \sum_{n=4}^{8} \epsilon^n \oint_{\partial B_A} dS_m \left[{}_n\Lambda_N^{\tau m} - v_A^m {}_n\Lambda_N^{\tau\tau} \right] y_A^{K_l i} = O(\epsilon^5) \tag{A1}$$

for $^\forall$ l and

$$R_1^{K_l ij} = \epsilon^{-4}\sum_{n=4}^{8}\epsilon^n \oint_{\partial B_1} dS_m \left[{}_n\Lambda_N^{jm} - v_1^m\, {}_n\Lambda_N^{j\tau}\right] y_1^{K_l i}$$
$$= \begin{cases} -\epsilon^4 \frac{3m_1^3 m_2}{5r_{12}^5} r_{12}^{<ij>} + O(\epsilon^5) & \text{for } l=0, \\ \epsilon^4 \frac{3m_1^3 m_2}{5r_{12}^3}\left(4\delta^{ki} r_{12}^j - \delta^{jk} r_{12}^i - \delta^{ij} r_{12}^k\right) + O(\epsilon^5) & \text{for } l=1, \\ O(\epsilon^5) & \text{otherwise.} \end{cases} \tag{A2}$$

$R_A^{K_l ij}$ contribute to the field at 3PN order through the moments $Z_A^{K_l ij}$. See Eqs. (3.21)-(3.30). Combining the above results, the $Q_A^{K_l i}$ and $R_A^{K_l ij}$ contributions to the 3PN field $h_{QR}^{\mu\nu}$ become

$${}_{10}h_{QR}^{\tau\tau} + {}_8h_{QRk}^{k} = 4\sum_{A=1,2}\frac{r_A^k}{r_A^3}\left({}_4R_A^{kll} - \frac{1}{2}{}_4R_A^{llk}\right) = C_{QR}\sum_{A=1,2}\frac{m_A^3(\vec{a}_A\cdot\vec{r}_A)}{r_A^3}, \tag{A3}$$

with $C_{QR}=12$ and $h_{QR}^{\tau i} = O(\epsilon^9)$, where we used Eqs. (A8), (A9), and (A10) below.

The $h_{QR}^{\mu\nu}$ field affects the equation of motion,

$$m_1 a_{1QR}^i = -\epsilon^6 \frac{C_{QR}}{2}\frac{m_1^3 m_2^2}{r_{12}^6} r_{12}^i - \epsilon^6\frac{C_{QR}}{2}\frac{m_1^2 m_2^3}{r_{12}^6} r_{12}^i, \tag{A4}$$

where a_{1QR}^i is the QR field contribution to the acceleration of the star 1. Note that in the above equation we have not yet taken into account the effect of the 3PN Q_A^i integral (${}_6Q_A^i$). ${}_6Q_A^i$ does not affect the 3PN field, but the 3PN equation of motion through the 3PN momentum-velocity relation. We evaluate ${}_6Q_A^i$ in Sec. VIII and Appendix E.

For convenience, we list $Q_A^{K_l i}$ integrals, $R_A^{K_l ij}$ integrals, and the body zone contribution including multipole moments up to 3PN order. We retain here (and only here) the dipole moments of order $O(\epsilon^2)$, since it is appropriate to define the representative points of the stars by $D_A^i = \epsilon^2 M_A^{ik} v_A^k + O(\epsilon^3)$ when we are concerned with the spin effects (see paper I).

$$Q_1^i = \epsilon^4 \frac{2m_2 M_1^{ik} n_{12}^k}{3r_{12}^2} + O(\epsilon^5), \tag{A5}$$

and $Q_1^{K_l i} = O(\epsilon^5)$ for $l \neq 0$,

$$\begin{aligned} R_1^{ji} &= \epsilon^4\frac{m_2}{3r_{12}^2}\left({}_2D_1^k n_{12}^k \delta^{ij} + 2\,{}_2D_1^j n_{12}^i - {}_2D_1^i n_{12}^j\right) \\ &+\epsilon^4 \frac{m_2}{3r_{12}^2}\Big(3\delta^{ij} M_1^{kl} n_{12}^k v_1^l - 3M_1^{ik} v_1^k n_{12}^j + 4M_1^{ik} n_{12}^k v_1^j - 4\delta^{ij} M_1^{kl} n_{12}^k v_2^l \\ &- 2M_1^{jk} v_2^k n_{12}^i + 2M_1^{jk} n_{12}^k v_2^i + 4M_1^{ik} v_2^k n_{12}^j - 4M_1^{ik} n_{12}^k v_2^j\Big) \\ &+\epsilon^4 \frac{m_2}{5r_{12}^3}\left(3I_1^{ij} - 3\delta^{ij} I_1^{kl} n_{12}^k n_{12}^l - 12 I_1^{jk} n_{12}^k n_{12}^i + 3I_1^{ik} n_{12}^k n_{12}^j + 3I_1{}^k{}_k n_{12}^i n_{12}^j\right) \\ &+\epsilon^4 \frac{2m_2}{5r_{12}^3}\Big(-2n_{12}^i n_{12}^j Z_1^{k[lk]l} - n_{12}^k n_{12}^j Z_1^{k[li]l} - n_{12}^k n_{12}^j Z_1^{i[lk]l} \\ &+ 4n_{12}^k n_{12}^i Z_1^{k[lj]l} + 4n_{12}^k n_{12}^i Z_1^{j[lk]l} + 2\delta^{ij} n_{12}^k n_{12}^l Z_1^{k[ml]m} \\ &- Z_1^{i[kj]k} - Z_1^{j[ki]k}\Big) + (\text{monopole part}) + O(\epsilon^5), \end{aligned} \tag{A6}$$

$$\begin{aligned} R_1^{jki} &= \epsilon^4 \frac{m_2}{5r_{12}^2}\left(2\delta^{i(j} I_1^{k)l} n_{12}^l - 2I_1^{i(j} n_{12}^{k)} - \delta^{jk} I_1{}^l{}_l n_{12}^i + 3I_1^{jk} n_{12}^i\right) \\ &+\epsilon^4\frac{2m_2}{15r_{12}^2}\Big(-2\delta^{jk} Z_1^{l[lm]m} n_{12}^i - \delta^{ij} Z_1^{l[mk]m} n_{12}^l - \delta^{ik} Z_1^{l[mj]m} n_{12}^l - \delta^{ij} Z_1^{k[ml]m} n_{12}^l \\ &- \delta^{ik} Z_1^{j[ml]m} n_{12}^l + Z_1^{i[lj]l} n_{12}^k + Z_1^{i[lk]l} n_{12}^j + Z_1^{j[li]l} n_{12}^k + Z_1^{k[li]l} n_{12}^j \\ &- 3Z_1^{j[lk]l} n_{12}^i - 3Z_1^{k[lj]l} n_{12}^i\Big) + (\text{monopole part}) + O(\epsilon^5), \end{aligned} \tag{A7}$$

and $R_1^{K_l ij} = O(\epsilon^5)$ for $l \neq 0, 1$.

In the following, we have used Eqs. (3.19)-(3.30):

$$\begin{aligned}
h_B^{\tau\tau} &= 4\epsilon^4 \sum_{A=1,2}\left[\frac{P_A^\tau}{r_A} + \epsilon^2\frac{D_A^k r_A^k}{r_A^3} + \epsilon^4\frac{3I_A^{kl} r_A^{<kl>}}{2r_A^5} + \epsilon^6\frac{5I_A^{klm} r_A^{<klm>}}{2r_A^7}\right] - 4\epsilon^5\frac{\partial}{\partial\tau}\sum_{A=1,2} P_A^\tau \\
&+2\epsilon^6\frac{\partial^2}{\partial\tau^2}\sum_{A=1,2}\left[P_A^\tau r_A - \epsilon^2\frac{r_A^k D_A^k}{r_A} + \epsilon^4\frac{\delta^{kl} - n_A^k n_A^l}{r_A} I_A^{kl}\right] \\
&-\frac{2}{3}\epsilon^7\frac{\partial^3}{\partial\tau^3}\sum_{A=1,2}\left[P_A^\tau r_A^2 - 2\epsilon^2 r_A^k D_A^k\right] + \frac{1}{6}\epsilon^8\frac{\partial^4}{\partial\tau^4}\sum_{A=1,2}\left[P_A^\tau r_A^3 - 3\epsilon^2 r_A r_A^k D_A^k\right] \\
&-\frac{1}{30}\epsilon^9\frac{\partial^5}{\partial\tau^5}\sum_{A=1,2}\left[P_A^\tau r_A^4\right] + \frac{1}{180}\epsilon^{10}\frac{\partial^6}{\partial\tau^6}\sum_{A=1,2}\left[P_A^\tau r_A^5\right] + O(\epsilon^{11}),
\end{aligned} \tag{A8}$$

$$\begin{aligned}
h_B^{\tau i} &= 4\epsilon^4 \sum_{A=1,2}\left[\frac{P_A^\tau v_A^i}{r_A} + \epsilon^2\left(\frac{1}{r_A}\frac{dD_A^i}{d\tau} + \frac{M_A^{ki} r_A^k}{2r_A^3} + \frac{v_A^{(k} D_A^{i)} r_A^k}{r_A^3}\right)\right. \\
&\left. + \epsilon^4\left(\frac{{}_4Q_A^i}{r_A} + \frac{r_A^k}{2r_A^3}\frac{dI_A^{ki}}{d\tau} + \frac{3v_A^{(i} I_A^{kl)} r_A^{<kl>}}{2r_A^5} + \frac{2J_A^{k[li]} r_A^{<kl>}}{r_A^5}\right)\right] \\
&-4\epsilon^5\frac{\partial}{\partial\tau}\sum_{A=1,2}\left[P_A^\tau v_A^i + \epsilon^2\frac{dD_A^i}{d\tau}\right] \\
&+2\epsilon^6\frac{\partial^2}{\partial\tau^2}\sum_{A=1,2}\left[P_A^\tau r_A v_A^i - \epsilon^2\left(\frac{r_A^k M_A^{ki}}{2r_A} + \frac{r_A^k v_A^{(k} D_A^{i)}}{r_A}\right)\right] \\
&-\frac{2}{3}\epsilon^7\frac{\partial^3}{\partial\tau^3}\sum_{A=1,2}\left[P_A^\tau r_A^2 v_A^i\right] + \frac{1}{6}\epsilon^8\frac{\partial^4}{\partial\tau^4}\sum_{A=1,2}\left[P_A^\tau r_A^3 v_A^i\right] + O(\epsilon^9),
\end{aligned} \tag{A9}$$

$$
\begin{aligned}
h_B^{ij} = 4\epsilon^4 \sum_{A=1,2} \Bigg[& \frac{P_A^\tau v_A^i v_A^j}{r_A} \\
& + \epsilon^2 \left(\frac{2v_A^{(i}}{r_A} \frac{dD_A^{j)}}{d\tau} + \frac{D_A^{(i}}{r_A} \frac{dv_A^{j)}}{d\tau} + \frac{M_A^{k(i} v_A^{j)} r_A^k}{r_A^3} + \frac{D_A^{(i} v_A^{j)} v_A^k r_A^k}{r_A^3} + \frac{Z_A^{k[li]j} r_A^{<kl>}}{r_A^5} + \frac{Z_A^{k[lj]i} r_A^{<kl>}}{r_A^5} \right) \\
& + \epsilon^4 \left(\frac{1}{2r_A} \frac{d^2 I_A^{ij}}{d\tau^2} + \frac{v_A^k r_A^k}{2r_A^3} \frac{dI_A^{ij}}{d\tau} + \frac{r_A^k}{2r_A^3} \frac{d}{d\tau} \left(v_A^{(k} I_A^{ij)} \right) - \frac{2r_A^k}{3r_A^3} \left(\frac{dJ_A^{i[jk]}}{d\tau} + \frac{dJ_A^{j[ik]}}{d\tau} \right) \right. \\
& + \frac{r_A^{<kl>}}{6r_A^5} \left(2v_A^k v_A^l I_A^{ij} + v_A^i v_A^j I_A^{kl} + 6 v_A^k v_A^{(i} I_A^{j)l} \right) \\
& + \frac{r_A^{<kl>}}{6r_A^5} \left(v_A^k J_A^{i[lj]} + v_A^k J_A^{j[li]} + 2v_A^i J_A^{k[lj]} + 2v_A^j J_A^{k[li]} \right) \\
& + \frac{15 Z_A^{kl[mi]j} r_A^{<klm>}}{8r_A^7} + \frac{15 Z_A^{kl[mj]i} r_A^{<klm>}}{8r_A^7} + \frac{{}_4Q_A^{(i} v_A^{j)}}{r_A} \\
& \left. + \frac{5 {}_4R_A^{klm(ij)} r_A^{<klm>}}{8r_A^7} + \frac{{}_4R_A^{(ij)}}{r_A} + \frac{r_A^k}{2r_A^3} ({}_4R_A^{kji} + {}_4R_A^{kij} - {}_4R_A^{ijk}) \right) \Bigg] \\
& -4\epsilon^5 \frac{\partial}{\partial\tau} \sum_{A=1,2} \left[P_A^\tau v_A^i v_A^j + \epsilon^2 \left(2v_A^{(i} \frac{dD_A^{j)}}{d\tau} + D_A^{(i} \frac{dv_A^{j)}}{d\tau} \right) \right] \\
& +2\epsilon^6 \frac{\partial^2}{\partial\tau^2} \sum_{A=1,2} \left[P_A^\tau v_A^i v_A^j r_A + \epsilon^2 \left(2r_A v_A^{(i} \frac{dD_A^{j)}}{d\tau} + r_A D_A^{(i} \frac{dv_A^{j)}}{d\tau} - \frac{M_A^{k(i} v_A^{j)} r_A^k}{r_A} \right. \right. \\
& \left. \left. - \frac{D_A^{(i} v_A^{j)} v_A^k r_A^k}{r_A} + \frac{\delta^{kl} - n_A^{kl}}{3r_A} Z_A^{k[li]j} + \frac{\delta^{kl} - n_A^{kl}}{3r_A} Z_A^{k[lj]i} \right) \right] \\
& -\frac{2}{3}\epsilon^7 \frac{\partial^3}{\partial\tau^3} \sum_{A=1,2} \left[P_A^\tau v_A^i v_A^j r_A^2 \right] + \frac{1}{6}\epsilon^8 \frac{\partial^4}{\partial\tau^4} \sum_{A=1,2} \left[P_A^\tau v_A^i v_A^j r_A^3 \right] \\
& + O(\epsilon^9). \qquad \text{(A10)}
\end{aligned}
$$

We restrict our attention to the equation of motion for two spherical compact stars in this paper, however our formulation can be extended to an extended body with higher multipole moments, as shown in paper I.

APPENDIX B: DERIVATION OF EQ. (3.31)

In this section, we show a derivation of Eq. (3.31) without using the Dirac delta distribution. The following proof is essentially due to [60].

Suppose that a two-dimensional surface S surrounds a three-dimensional volume V. Suppose that there is a point $P(\vec{x})$ and a point $Q(\vec{y})$ both inside of V, and the distance between the two points is r. Thus,

$$r = |\vec{x} - \vec{y}|.$$

Note that $\nabla^2(1/r) = 0$ except for $r = 0$.

Define a sphere V' which is centered at $P(\vec{x})$ and has a radius d that is sufficiently small so that V' is enclosed completely by V. The surface Σ of V' divides V into two regions. We call the outer region V''.

The Green's theorem (e.g., [61]) states for a certain function $v = v(\vec{x})$ that

$$
\begin{aligned}
\int_{V''} \frac{\nabla^2 v(\vec{y})}{r} d^3y + \oint_S \left\{ \frac{1}{r} \frac{\partial v(\vec{y})}{\partial y^i} - v(\vec{y}) \frac{\partial}{\partial y^i} \left(\frac{1}{r} \right) \right\} dS_i \\
+ \oint_\Sigma \left\{ \frac{1}{r} \frac{\partial v(\vec{y})}{\partial y^i} - v(\vec{y}) \frac{\partial}{\partial y^i} \left(\frac{1}{r} \right) \right\} dS_i = 0. \qquad \text{(B1)}
\end{aligned}
$$

Note that V'' does not include the point $P(\vec{x})$. We assume here that for $v(\vec{x})$ the second derivative with respect to $\vec{x}$ exists and is continuous in V''. In the end of this section, we mention whether this assumption holds for our particular example, Eq. (3.31).

The third term can be evaluated as

$$\oint_{\Sigma}\left\{\frac{1}{r}\frac{\partial v(\vec{y})}{\partial y^i}-v(\vec{y})\frac{\partial}{\partial y^i}\left(\frac{1}{r}\right)\right\}dS_i=d\oint_{\Sigma}\frac{\partial v(\vec{y})}{\partial y^i}n_i d\Omega+\oint_{\Sigma}v(\vec{y})d\Omega, \tag{B2}$$

where n_i is the outward normal of the surface Σ. If $\partial v(\vec{y})/\partial y^i$ is finite in the $d\to 0$ limit, then the first term is zero in this limit. Thus,

$$\oint_{\Sigma}\left\{\frac{1}{r}\frac{\partial v(\vec{y})}{\partial y^i}-v(\vec{y})\frac{\partial}{\partial y^i}\left(\frac{1}{r}\right)\right\}dS_i=4\pi v(\vec{x}), \tag{B3}$$

in the $d\to 0$ limit.

In the same manner,

$$\lim_{d\to 0}\int_{V'}\frac{\nabla^2 v(\vec{y})}{r}d^3y=\lim_{d\to 0}\int_{V'}\nabla^2 v(\vec{y})rdrd\Omega=0. \tag{B4}$$

Then we have

$$\int_{V}\frac{\nabla^2 v(\vec{y})}{r}d^3y+\oint_{S}\left\{\frac{1}{r}\frac{\partial v(\vec{y})}{\partial y^i}-v(\vec{y})\frac{\partial}{\partial y^i}\left(\frac{1}{r}\right)\right\}dS_i+4\pi v(\vec{x})=0. \tag{B5}$$

This is essentially Eq. (3.31).

In the particular case of Eq. (3.31), $V=N/B$. We recall that the stars are completely enclosed by the body zones B_A and thus there exists no matter in N/B. The integrand $v(\vec{x})$ or $f(x)$ and $g(x)$ in Eq. (3.31) have thus no singularity in N/B even in the point-particle limit. These functions have a smooth second derivative with respect to $\vec{x}$ in N/B.

APPENDIX C: CORRECTION TO THE MULTIPOLE MOMENTS

A natural reference coordinate where we would define multipole moments of a star may be a coordinate in which effects of its orbital motion and the companion star are removed (modulo, namely, the tidal effect). In other words, such a natural reference coordinate may be the generalized Fermi coordinate [59], in which the metric is Lorentzian at $z_A^i(\tau)$. We define the two stars to be spherical in such a coordinate in this paper [30].

In paper II, we defined a multipole moment, which we call the NZC moment in this section, as a volume integral over the body zone which is spherical in the near zone coordinate (NZC). Then a question specific to our formalism is whether the NZC moments affect the orbital motion of the stars which are spherical in the generalized Fermi coordinate (GFC).

At lowest order ($O(\epsilon^0)$), the NZC multipole moments and the GFC multipole moments defined as volume integrals over a sphere in GFC must be the same. For a spherically symmetric star, the NZC moments and the GFC moments with trace-free operation or antisymmetrization on their indexes vanish at the lowest order. We further assume that the GFC moments and the NZC moments are zero for a spherical compact star at the lowest order because of its compactness.

For the mass monopole, we have already obtained the relation between the NZC monopole (the energy) and the mass via the evolution equation of the energy. Thus, we seek the 1PN correction to the NZC Lth multipole moments with $L\geq 2$ since physically relevant multipole moments appear at 2PN order and we are concerned with the 3PN equation of motion in this paper. In the following, we compute the corrections which do not include higher order multipole moments than the energy monopole.

As for construction of GFC, we assume, up to the relevant order here, that the transformation from NZC to GFC or vice versa takes apparently the same form for a strongly self-gravitating star in which we are interested here and for a weakly self-gravitating star for which GFC has been constructed in [59]. This is because the construction of the generalized Fermi coordinate depends on how the star moves, and because the equations of motion for binary stars take the same form regardless of the strength of the stars' internal gravity up to 2.5PN order (paper II). An important difference is that the mass parameter in the transformation from NZC to GFC for the strongly self-gravitating star includes the strong field effect, as explained below Eq. (3.36).

We now assume the following coordinate transformations from GFC $(\hat{\tau},\hat{x}^k)$ to NZC (τ,x^k):

$$y^i=z_A^i(\tau)+\hat{y}_A^i+\delta y^i(\tau,\hat{y}_A^k), \tag{C1}$$

$$\tau-\tau_P=\hat{\tau}-\hat{\tau}_P+\delta\tau(\tau,\hat{y}_A^i), \tag{C2}$$

with

$$\delta y^i(\tau, \hat{y}_A^k) = \sum_{n=1} A^{iJ_n}(\tau)\hat{y}_A^{J_n}, \tag{C3}$$

$$\delta\tau(\tau, \hat{y}_A^k) = \sum_{n=0} B^{J_n}(\tau)\hat{y}_A^{J_n}, \tag{C4}$$

where the capital index denotes a set of collective indexes, e.g., $A^{iJ_n} = A^{ij_1\cdots j_n}$. $\hat{y}_A^i = \hat{y}^i - \hat{z}_A^i$. In Eq. (C1), $\hat{y}_A^i$ is on the $\hat{\tau} =$ const surface. τ_P and $\hat{\tau}_P$ are fiducial time coordinates. We assume that $A^{iJ_n}, B^{J_n} = O(\epsilon^2)$.

The NZC moments are defined on $B_A(\tau_P)$ which is a sphere centered at $(\tau_P, z_A^i(\tau_P))$ with radius ϵR_A in NZC, while the GFC moments are defined on $\hat{B}_A(\hat{\tau}_P)$ which is centered at the same event $(\hat{\tau}_P, \hat{z}_A^i)$ in GFC ($(\tau_P, z_A^i(\tau_P))$ in NZC) with the radius ϵR_A in GFC. The corrections we shall compute are then

$$\epsilon^{2l+4-s}\delta I_A^{J_l\mu\nu} \equiv \epsilon^{2l+4-s} I_{A,\mathrm{NZC}}^{J_l\mu\nu} - \epsilon^{2l+4-s} I_{A,\mathrm{GFC}}^{J_l\mu\nu}, \tag{C5}$$

$$\epsilon^{2l+4-s} I_{A,\mathrm{NZC}}^{J_l\mu\nu} \equiv \int_{B_A(\tau=\tau_P)} d^3y_A y_A^{J_l}(\tau)\Lambda_N^{\mu\nu}(\tau, y^i), \tag{C6}$$

$$\epsilon^{2l+4-s} I_{A,\mathrm{GFC}}^{J_l\mu\nu} \equiv \int_{\hat{B}_A(\hat{\tau}=\hat{\tau}_P)} d^3\hat{y}_A \hat{y}_A^{J_l}\hat{\Lambda}_{G'}^{\mu\nu}(\hat{\tau}, \hat{y}^i), \tag{C7}$$

where $s = 2$ for $(\mu, \nu) = (i, j)$ or 0 otherwise. $\hat{\Lambda}_{G'}^{\mu\nu}(\hat{\tau}, \hat{y}^i) = \hat{\Lambda}_G^{\mu\nu}(\hat{\tau}, \hat{y}_A^i)$.

We now express $I_{A,\mathrm{NZC}}^{J_l\mu\nu}$ using the generalized Fermi coordinate. Note that $y_A^{J_l}(\tau)$ in Eq. (C6) is on the $\tau = \tau_P =$ const surface. A necessary coordinate transformation that relates $y_A^i(\tau)$ on the $\tau =$ const surface with $\hat{y}_A^i$ on the $\hat{\tau} =$ const surface can be obtained by modifying Eq. (C1) using retardation expansion. Up to 1PN order, the result is

$$y^i = z_A^i(\tau) + \hat{y}_A^i - \tilde{\delta} y^i(\tau_P, \hat{y}_A^k), \tag{C8}$$

with

$$\tilde{\delta} y^i(\tau_P, \hat{y}_A^k) = \sum_{n=1}\left(B^{J_n}(\tau_P)v_A^i(\tau_P) - A^{iJ_n}(\tau_P)\right)\hat{y}_A^{J_n}. \tag{C9}$$

Using tetrad $e^\mu{}_{\hat{\alpha}}(\hat{\tau}, \hat{y}^i) = \delta^\mu{}_{\hat{\alpha}} + O(\epsilon^2)$, we have

$$\begin{aligned}
\epsilon^{2l+4-s} I_{A,\mathrm{NZC}}^{J_l\mu\nu} &= \int_{\tilde{B}_A\left(\hat{\tau}=\hat{\tau}(\hat{\tau}_P, \hat{y}_A^i)\right)} d^3\hat{y}_A \left|\frac{\partial(y_A^i)}{\partial(\hat{y}_A^j)}\right| \\
&\quad\times \prod_{k=1}^{l}\left\{\hat{y}_A^{j_k} - \delta\tilde{y}^{j_k}(\tau, \hat{y}_A^i)\right\} \\
&\quad\times e^\mu{}_{\hat{\alpha}}(\hat{\tau}, \hat{y}_A^i)e^\nu{}_{\hat{\beta}}(\hat{\tau}, \hat{y}_A^i)\hat{\Lambda}_{G'}^{\alpha\beta}(\hat{\tau}, \hat{y}^i) \\
&= \int_{\tilde{B}_A\left(\hat{\tau}=\hat{\tau}(\hat{\tau}_P, \hat{y}_A^i)\right)} d^3\hat{y}_A \hat{y}_A^{J_l}\hat{\Lambda}_{G'}^{\mu\nu}(\hat{\tau}, \hat{y}^i),
\end{aligned} \tag{C10}$$

where in the last equality we neglected all the corrections that result in the multipole moments of order $L' \geq L$. The integral region $\tilde{B}_A\left(\hat{\tau} = \hat{\tau}(\hat{\tau}_P, \hat{y}_A^i)\right)$ which corresponds to $B_A(\tau = \tau_P = \mathrm{const})$ is not a $\hat{\tau} = \hat{\tau}_P$ const 3-surface nor spherical in GFC. In fact, from Eq. (C2),

$$\hat{\tau}(\hat{\tau}_P, \hat{y}_A^i) = \hat{\tau}_P - \delta\tau(\tau_P, \hat{y}_A^i). \tag{C11}$$

Thus we make a retardation expansion around $\hat{\tau} = \hat{\tau}_P$ in the integrand. Then to make the slightly perturbed sphere $\tilde{B}_A(\hat{\tau} = \hat{\tau}_P)$ into the sphere $\hat{B}_A(\hat{\tau} = \hat{\tau}_P)$ with radius ϵR_A, we change the integration variable $\hat{y}_A^i$ into $\check{y}_A^i$. Up to 1PN order, the transformation may be obtained by inverting Eq. (C8),

$$\hat{y}^i = \hat{z}_A^i(\hat{\tau}_P) + \check{y}_A^i + \tilde{\delta} y^i(\tau_P, \check{y}_A^k). \tag{C12}$$

Using Eq. (C11) and Eq. (C12), we simplify Eq. (C10) up to 1PN order as

$$
\begin{aligned}
&\epsilon^{2l+4-s} I^{J_l\mu\nu}_{A,\mathrm{NZC}} \\
&= \int_{\tilde{B}_A(\hat{\tau}_P)} d^3\hat{y}_A \hat{y}_A^{J_l} \left\{ \hat{\Lambda}^{\mu\nu}_{G'}\left(\hat{\tau}_P - \delta\tau(\tau_P, \hat{y}^i_A), \hat{y}^i\right)\right\} \\
&= \int_{\tilde{B}_A(\hat{\tau}_P)} d^3\hat{y}_A \hat{y}_A^{J_l} \left\{ \hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \hat{y}^i) - \delta\tau(\tau_P, \hat{y}^i_A)\frac{\partial}{\partial\hat{\tau}_P}\hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \hat{y}^i)\right\} \\
&= \int_{\hat{B}_A(\hat{\tau}_P)} d^3\check{y}_A \left|\frac{\partial(\hat{y}^i_A)}{\partial(\check{y}^j_A)}\right| \prod_{k=1}^{l}\left\{\check{y}^{j_k}_A + \tilde{\delta}y^{j_k}(\tau_P, \check{y}^i_A)\right\} \\
&\quad\times\left\{\hat{\Lambda}^{\mu\nu}_{G}\left(\hat{\tau}_P, \check{y}^i_A + \tilde{\delta}y^i(\tau_P, \check{y}^i_A)\right) - \delta\tau(\tau_P, \check{y}^i_A)\frac{\partial}{\partial\hat{\tau}_P}\hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \check{y}^i)\right\} \\
&= \int_{\hat{B}_A(\hat{\tau}_P)} d^3\check{y}_A \left\{\check{y}^{J_l}_A \hat{\Lambda}^{\mu\nu}_{G'}\left(\hat{\tau}_P, \check{y}^i\right) + \tilde{\delta}y^m(\tau_P, \check{y}^i_A)\frac{\partial}{\partial\check{y}^m}\left(\check{y}^{J_l}_A\hat{\Lambda}^{\mu\nu}_{G'}\left(\hat{\tau}_P, \check{y}^i\right)\right)\right. \\
&\quad\left. - \check{y}^{J_l}_A\delta\tau(\tau_P, \check{y}^i_A)\frac{\partial}{\partial\hat{\tau}_P}\hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \check{y}^i)\right\}. \qquad \text{(C13)}
\end{aligned}
$$

As before, we have discarded terms which end up with multipole moments of order $L' \geq L$. Integrating by parts and rewriting $\check{y}^i$ by $\hat{y}^i$, we finally obtain a formula for $\delta I^{J_l\mu\nu}_A$,

$$
\begin{aligned}
&\epsilon^{2l+4-s}\delta I^{J_l\mu\nu}_A \\
&= \sum_{n=1}\left(B^{K_n}(\tau_P)v^m_A(\tau_P) - A^{mK_n}(\tau_P)\right)\oint_{\partial\hat{B}_A(\hat{\tau}_P)} d\hat{S}_m \hat{y}^{J_l}_A\hat{y}^{K_n}_A\hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \hat{y}^i) \\
&\quad - \sum_{n=1} B^{K_n}(\tau_P)\frac{d}{d\hat{\tau}_P}\int_{\hat{B}_A(\hat{\tau}_P)} d^3\hat{y}_A\hat{y}^{J_l}_A\hat{y}^{K_n}_A\hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \hat{y}^i) \\
&\quad - \sum_{n=1}\left(B^{K_n}(\tau_P)v^m_A(\tau_P) - A^{mK_n}(\tau_P)\right) \\
&\quad \times\int_{\hat{B}_A(\hat{\tau}_P)} d^3\hat{y}_A\frac{\partial\hat{y}^{K_n}_A}{\partial\hat{y}^m}\hat{y}^{J_l}_A\hat{\Lambda}^{\mu\nu}_{G'}(\hat{\tau}_P, \hat{y}^i). \qquad \text{(C14)}
\end{aligned}
$$

Notice that $d\hat{z}^i_A(\hat{\tau}_P)/d\hat{\tau}_P = 0$.

The last two volume integrals in the above equation result in the multipole moments and we neglect them here. As for the first term, we can evaluate the surface integral to 1PN order explicitly by replacing all the hatted quantities by those without a hat (and G' by N). We found that ${}_n\Lambda^{\mu\nu}_N$ ($n \leq 7$) do not contribute for any L. On the other hand, ${}_8\Lambda^{\tau\tau}_N$ for $L = 2$ and $n = 1$ in the summation in the first term of Eq. (C14) gives a monopole correction to the 3PN gravitational field (through the 1PN correction to the quadrupole moment which itself appears at 2PN order in the field),

$$
\begin{aligned}
\oint_{\partial\hat{B}_A(\hat{\tau}_P)} d\hat{S}_m\hat{y}^i_A\hat{y}^j_A\hat{y}^k_A\hat{\Lambda}^{\tau\tau}_{G'}(\hat{\tau}_P, \hat{y}^i) &= \oint_{\partial B_A(\tau_P)} dS_m y^i_A y^j_A y^k_A\Lambda^{\tau\tau}_N(\tau_P, y^i)(1 + O(\epsilon^2)) \\
&= \epsilon^8\oint_{\partial B_A(\tau_P)} dS_m y^i_A y^j_A y^k_A {}_8\Lambda^{\tau\tau}_N(\tau_P, y^i) + O(\epsilon^{10}) \\
&= -\epsilon^8\frac{4m^3_A}{5}\left(\delta^{ij}\delta^k_m + \delta^{kj}\delta^i_m + \delta^{ik}\delta^j_m\right). \qquad \text{(C15)}
\end{aligned}
$$

Neither ${}_8\Lambda^{\mu\nu}_N$ for $L \geq 3$ nor $n \geq 2$ contribute to the 3PN field.

The coefficients $A^{mk}(\tau_P)$ and $B^k(\tau_P)$ may be read off from the results in [59,62], which are

$$B^i(\tau_P)v^j_1(\tau_P) = \epsilon^2 v^i_1 v^j_1 + O(\epsilon^3) \qquad \text{(C16)}$$

$$A^{ij}(\tau_P) = \epsilon^2\left(\frac{1}{2}v^i_1 v^j_1 - \frac{m_2}{r_{12}}\delta^{ij}\right) + O(\epsilon^3) \qquad \text{(C17)}$$

for the star $A = 1$. Thus, the coefficient in front of the surface integral, Eq. (C15), becomes $\epsilon^2(v^i_1 v^j_1/2 + \delta^{ij}m_2/r_{12})$.

Finally using the coefficients above, we obtain the 1PN corrections of the multipole moments that affect a 3PN equation of motion for spherical stars (we rename $\delta I^{ij\tau\tau}_A$ by δI^{ij}_A),

$$\delta I_1^{ij} = \epsilon^2\left(-\frac{4m_1^3}{5}v_1^i v_1^j - \frac{2m_1^3 v_1^2}{5}\delta^{ij} - \frac{4m_1^3 m_2}{r_{12}}\delta^{ij}\right) + O(\epsilon^3). \tag{C18}$$

A similar equation holds for the star $A = 2$ by exchanging $1 \leftrightarrow 2$ in the above equation. Notice that since the quadrupole moments at 2PN order is symmetric-trace-free, the last two terms do not contribute to the 3PN field.

The correction $\delta I_1^{<ij>}$ should appear even when $m_2 \to 0$. Not surprisingly, with the correction $\delta I_1^{<ij>}$, the 3PN gravitational field for a single star moving at a constant velocity derived by solving the harmonically relaxed Einstein equations iteratively agrees with the boosted Schwarzschild metric in the harmonic coordinate up to 3PN order.

It is possible to use Eq. (C14) for the $L = 1$ case and the result becomes again the 3PN correction to the field (we again rename $\delta I_1^{i\tau\tau}$ as δD_1^i),

$$\delta D_1^i = \epsilon^4 \frac{2m_1^3 m_2}{r_{12}^3} r_{12}^i. \tag{C19}$$

(In the computation of δD_1^i, the surface integral with ${}_8\chi_N^{\tau\tau\alpha\beta}{}_{,\alpha\beta}$ as the integrand is found to vanish.) However, any change of the dipole moment amounts merely to a redefinition of the representative point of the star and it causes no physical effect. In fact, we have not taken into account δD_1^i to derive our 3PN equation of motion, Eq. (11.4).

APPENDIX D: RENORMALIZATION OF THE MULTIPOLE MOMENTS

This section explains the "renormalization" of the multipole moments and that the field does not depend on ϵR_A. This is rather trivial, however we show this section for clarity.

We first define a symbol $\text{part}_{\epsilon R_A} F$ which is the ϵR_A-dependent part in an expression F except for the logarithmic dependence of ϵR_A. Correspondingly, we define $\text{disc}_{\epsilon R_A} F$, which means to discard all the ϵR_A-dependent terms in F other than $\ln \epsilon R_A$. By construction,

$$F = \operatorname*{disc}_{\epsilon R_A} F + \operatorname*{part}_{\epsilon R_A} F. \tag{D1}$$

We give an example of $\text{disc}_{\epsilon R_A}$ in Sec. V C.

Now, to derive the field, we study the following Poisson integral for a certain function $f(\vec{x})$. $f(\vec{x})$ is some combination of $\Lambda^{\mu\nu}(\tau, \vec{x})$ and is assumed to be nonsingular in N. For notational simplicity, we do not write time dependence explicitly in $f(\vec{x})$,

$$\int_N \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) = \sum_{A=1,2} \int_{B_A} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) + \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}). \tag{D2}$$

The second volume integral is evaluated with the help of the superpotential $g(\vec{x})$ that satisfies $\Delta g(\vec{x}) = f(\vec{y})$ in N/B. Using Eq. (3.31) and expanding the kernel $1/|\vec{x}-\vec{y}|$ around $\vec{y}_A = \vec{y} - \vec{z}_A$, we obtain

$$\begin{aligned}\int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) = &- \sum_{A=1,2}\sum_{n=0} \frac{(2n-1)!!r_A^{<K_n>}}{n!r_A^{2n+1}} \oint_{B_A} dS_k y_A^{K_n} \frac{\partial g(\vec{y}_A + \vec{z}_A)}{\partial y_A^k} \\ &+ \sum_{A=1,2}\sum_{n=0} \frac{(2n-1)!!r_A^{<K_n>}}{n!r_A^{2n+1}} \oint_{B_A} dS_k g(\vec{y}_A + \vec{z}_A) \frac{\partial y_A^{K_n}}{\partial y_A^k} \\ &+ \cdots . \end{aligned} \tag{D3}$$

Here we only show explicitly the terms which possibly depend on ϵR_A, and "$\cdots$" denotes ϵR_A-independent terms. Thus, using the symbols introduced above, we have

$$\int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) = \operatorname*{disc}_{\epsilon R_A} \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) + \operatorname*{part}_{\epsilon R_A} \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}), \tag{D4}$$

with

$$
\begin{aligned}
&\operatorname*{part}_{\epsilon R_A} \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) \\
&= \operatorname*{part}_{\epsilon R_A} \left[- \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!!r_A^{K_n}}{n! r_A^{2n+1}} \oint_{B_A} dS_k y_A^{K_n} \frac{\partial g(\vec{y}_A + \vec{z}_A)}{\partial y_A^k} \right. \\
&\left. + \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!!r_A^{K_n}}{n! r_A^{n+1}} \oint_{B_A} dS_k g(\vec{y}_A + \vec{z}_A) \frac{\partial y_A^{K_n}}{\partial y_A^k} \right].
\end{aligned} \tag{D5}
$$

For the first volume integral in Eq. (D2), we use the multipole expansion.

$$
\sum_{A=1,2} \int_{B_A} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) = \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!!r_A^{<K_n>} I_A^{K_n}}{n! r_A^{2n+1}}, \tag{D6}
$$

with

$$
I_A^{K_n} \equiv \int_{B_A} d^3y_A f(\vec{y}_A + \vec{z}_A) y_A^{K_n}. \tag{D7}
$$

$I_A^{K_n}$ is the multipole moment of the star A. We simplify here the definition of the multipole moments so that we omit the scalings on the integrand $\Lambda^{\mu\nu}(\tau, \vec{x})$ and hence $f(\vec{x})$. Obviously, this definition is enough to study the ϵR_A dependence in the multipole moments and the field.

$I_A^{K_n}$ in general depends on ϵR_A since the integrand $f(\vec{y})$ is noncompact support. A possible ϵR_A dependence in $I_A^{K_n}$ may be examined by the following. First, since we are studying the nonsingular sources, we expect that $f(\vec{x})$ is smooth so that we can assume $\Delta g(\vec{x}) = f(\vec{x})$ just inside of ∂B_A. Thus, the ϵR_A dependence in $I_A^{K_n}$ can be examined via

$$
\begin{aligned}
I_A^{K_n} &= \int_{\text{in the neighborhood of } \partial B_A} d^3y_A \Delta g(\vec{y}_A + \vec{z}_A) y_A^{K_n} + \cdots \\
&= \oint_{\partial B_A} dS_k \left(y_A^{K_n} \frac{\partial g(\vec{y}_A + \vec{z}_A)}{\partial y_A^k} - \frac{\partial y_A^{K_n}}{\partial y_A^k} g(\vec{y}_A + \vec{z}_A) \right) \\
&+ \cdots .
\end{aligned} \tag{D8}
$$

Here "$\cdots$" again denotes terms that do not depend on ϵR_A.

Using the symbols introduced in this section, we have

$$
I_A^{K_n} = \operatorname*{disc}_{\epsilon R_A} I_A^{K_n} + \operatorname*{part}_{\epsilon R_A} I_A^{K_n}, \tag{D9}
$$

$$
\operatorname*{disc}_{\epsilon R_A} I_A^{K_n} = \operatorname*{disc}_{\epsilon R_A} \int_{B_A} d^3y_A f(\vec{y}_A + \vec{z}_A) y_A^{K_n}, \tag{D10}
$$

$$
\begin{aligned}
\operatorname*{part}_{\epsilon R_A} I_A^{K_n} &= \operatorname*{part}_{\epsilon R_A} \int_{B_A} d^3y_A f(\vec{y}_A + \vec{z}_A) y_A^{K_n} \\
&= \operatorname*{part}_{\epsilon R_A} \left[\oint_{\partial B_A} dS_k \left(y_A^{K_n} \frac{\partial g(\vec{y}_A + \vec{z}_A)}{\partial y_A^k} - \frac{\partial y_A^{K_n}}{\partial y_A^k} g(\vec{y}_A + \vec{z}_A) \right) \right].
\end{aligned} \tag{D11}
$$

We call $I_{A,r}^{K_n} \equiv \text{disc}_{\epsilon R_A} I_A^{K_n}$ the "renormalized" multipole moments that are independent of powers of ϵR_A by definition of $\text{disc}_{\epsilon R_A}$.

It is clear that the ϵR_A dependences in the multipole moments Eq. (D11) cancel out those of the N/B contribution shown as the first and the second terms in Eq. (D3). In conclusion, we find that we can discard (or neglect) all the ϵR_A-independent terms other than $\ln \epsilon R_A$ terms when we compute the field,

$$
\begin{aligned}
\int_N \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) &= \sum_{A=1,2} \int_{B_A} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) + \int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|} f(\vec{y}) \\
&= \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!! r_A^{<K_n>} I_A^{K_n}}{n! r_A^{2n+1}} + \int_{N/B} \frac{d^3y f(\vec{y})}{|\vec{x}-\vec{y}|} \\
&= \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!! r_A^{<K_n>} I_{A,r}^{K_n}}{n! r_A^{2n+1}} + \operatorname*{disc}_{\epsilon R_A} \int_{N/B} \frac{d^3y f(\vec{y})}{|\vec{x}-\vec{y}|} \\
&+ \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!! r_A^{<K_n>}}{n! r_A^{2n+1}} \operatorname*{part}_{\epsilon R_A} I_A^{K_n} + \operatorname*{part}_{\epsilon R_A} \int_{N/B} \frac{d^3y f(\vec{y})}{|\vec{x}-\vec{y}|} \\
&= \sum_{A=1,2} \sum_{n=0} \frac{(2n-1)!! r_A^{<K_n>} I_{A,r}^{K_n}}{n! r_A^{2n+1}} + \operatorname*{disc}_{\epsilon R_A} \int_{N/B} \frac{d^3y f(\vec{y})}{|\vec{x}-\vec{y}|}.
\end{aligned} \tag{D12}
$$

In the main body of this paper other than this section, we write $I_{A,r}^{K_n}$ as $I_A^{K_n}$ and omit the symbol $\mathrm{disc}_{\epsilon R_A}$ in front of the Poisson integral over N/B for notational simplicity and from triviality of the fact that the total field is independent of ϵR_A. As an exception, we write $\mathrm{disc}_{\epsilon R_A}$ in Secs. V and IX to make our discarding ϵR_A procedure clear.

Finally, we mention that it is straightforward to extend the arguments here to show that the cancellation of ϵR_A terms between the body zone contribution and the N/B contribution occurs for any retarded field, that is, $n \geq 1$ terms in Eq. (3.10).

APPENDIX E: χ PART

We derive the functional expressions of $P^\mu_{A\chi}$ on m_A, v^i_A, and r^i_{12}. Here we defined $P^\mu_{A\chi}$ as

$$
P^\mu_{A\chi} \equiv \epsilon^{-4} \int_{B_A} d^3y \chi_N^{\mu\tau\alpha\beta}{}_{,\alpha\beta}. \tag{E1}
$$

By the definition of $\chi_N^{\mu\nu\alpha\beta}{}_{,\alpha\beta}$,

$$
16\pi \chi_N^{\tau\tau\alpha\beta}{}_{,\alpha\beta} = (h^{\tau k} h^{\tau l} - h^{\tau\tau} h^{kl})_{,kl},
$$

$$
16\pi \chi_N^{\tau i\alpha\beta}{}_{,\alpha\beta} = (h^{\tau\tau} h^{ik} - h^{\tau i} h^{\tau k})_{,\tau k} + (h^{\tau k} h^{il} - h^{\tau i} h^{kl})_{,kl},
$$

thus we can obtain the functional expressions of $P^\mu_{A\chi}$ using Gauss' law. In fact, up to 3PN order, the definition of $P^\tau_{A\chi}$ gives

$$
\begin{aligned}
P^\tau_{1\chi} &= \epsilon^4 \frac{m_1 m_2}{3 r_{12}} \left[4V^2 + \frac{m_2}{r_{12}} - \frac{2m_1}{r_{12}} \right] \\
&- \epsilon^5 \frac{2}{3} m_1 {}^{(3)}I^k_{\mathrm{orb}\, k} \\
&+ \epsilon^6 \frac{m_1 m_2}{r_{12}} \left[-\frac{2m_1^2}{3r_{12}^2} - \frac{5m_1 m_2}{r_{12}^2} + \frac{m_2^2}{r_{12}^2} + \frac{m_1}{r_{12}} \left(\frac{14}{5} v_1^2 + \frac{11}{3} v_2^2 \right. \right. \\
&- \left. \frac{22}{3} (\vec{v}_1 \cdot \vec{v}_2) - \frac{2}{5} (\vec{n}_{12} \cdot \vec{v}_1)^2 + \frac{20}{3} (\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) - 4 (\vec{n}_{12} \cdot \vec{v}_2)^2 \right) \\
&+ \frac{m_2}{r_{12}} \left(\frac{197}{30} v_1^2 + \frac{19}{3} v_2^2 - \frac{38}{3} (\vec{v}_1 \cdot \vec{v}_2) + \frac{2}{15} (\vec{n}_{12} \cdot \vec{v}_1)^2 - \frac{2}{3} (\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) \right) \\
&+ \frac{22}{15} v_1^4 + \frac{34}{15} v_1^2 v_2^2 + \frac{4}{3} v_2^4 - \frac{44}{15} v_1^2 (\vec{v}_1 \cdot \vec{v}_2) - \frac{8}{3} v_2^2 (\vec{v}_1 \cdot \vec{v}_2) + \frac{8}{15} (\vec{v}_1 \cdot \vec{v}_2)^2 \\
&- \left. \frac{2}{3} v_1^2 (\vec{n}_{12} \cdot \vec{v}_2)^2 - \frac{2}{3} v_2^2 (\vec{n}_{12} \cdot \vec{v}_2)^2 + \frac{4}{3} (\vec{v}_1 \cdot \vec{v}_2)(\vec{n}_{12} \cdot \vec{v}_2)^2 \right] \\
&+ O(\epsilon^7).
\end{aligned} \tag{E2}
$$

$P^\tau_{A\chi}$ of $O(\epsilon^6)$ affects a 3PN equation of motion only through the field ${}_{10}h^{\tau\tau}$.

The dipole moment and Q^i_A integral of the χ part give a nonzero contribution starting from 3PN order in the momentum-velocity relation,

$$D^i_{1\chi} = \epsilon^4 \frac{175 m_1^3 m_2}{18 r_{12}^3} r^i_{12} + O(\epsilon^5), \tag{E3}$$

$$Q^i_{1\chi} = \epsilon^6 \frac{m_1^3 m_2}{6 r_{12}^3} \left(-\frac{73 n_{12}^{<ij>} v_1^j}{5} + 11 n_{12}^{<ij>} v_2^j \right) + O(\epsilon^7). \tag{E4}$$

Here again, we used Gauss law to derive $D^i_{1\chi}$. On the other hand, we evaluate $P^i_{A\chi}$ directly from the definition of the $P^i_{A\chi}$ using Gauss law and found that

$$P^i_{A\chi} = P^\tau_{A\chi} v^i_A + Q^i_{A\chi} + \epsilon^2 \frac{dD^i_{A\chi}}{d\tau} + O(\epsilon^7) \tag{E5}$$

is an identity up to 3PN order. Thus, the representative point of the star is defined with the Θ part of the momentum-velocity relation, not with the χ part.

Finally, by evaluating the surface integrals in the evolution equation,

$$\frac{dP^\mu_{A\chi}}{d\tau} = -\epsilon^{-4} \oint_{\partial B_A} dS_k \chi_N^{\mu k\alpha\beta}{}_{,\alpha\beta} + \epsilon^{-4} v^k_A \oint_{\partial B_A} dS_k \chi_N^{\mu\tau\alpha\beta}{}_{,\alpha\beta}, \tag{E6}$$

we found that the resulting equations for $dP^\mu_{A\chi}/d\tau$ are consistent with the explicit expressions of $P^\mu_{A\chi}$ directly obtained from their definitions, Eqs. (E2) and (E5), as expected.

APPENDIX F: LANDAU-LIFSHITZ PSEUDOTENSOR EXPANDED IN EPSILON

The Landau-Lifshitz pseudotensor [53] in terms of $h^{\mu\nu}$ which satisfies the harmonic condition is as follows.

$$\begin{aligned}(-16\pi g) t^{\mu\nu}_{LL} &= g_{\alpha\beta} g^{\gamma\delta} h^{\mu\alpha}{}_{,\gamma} h^{\nu\beta}{}_{,\delta} + \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} h^{\alpha\gamma}{}_{,\delta} h^{\beta\delta}{}_{,\gamma} - 2 g_{\alpha\beta} g^{\gamma(\mu} h^{\nu)\alpha}{}_{,\delta} h^{\delta\beta}{}_{,\gamma} \\ &\quad + \frac{1}{2}\left(g^{\mu\alpha} g^{\nu\beta} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \right)\left(g_{\gamma\delta} g_{\epsilon\zeta} - \frac{1}{2} g_{\gamma\epsilon} g_{\delta\zeta} \right) h^{\gamma\epsilon}{}_{,\alpha} h^{\delta\zeta}{}_{,\beta}.\end{aligned} \tag{F1}$$

We expand the deviation field $h^{\mu\nu}$ in a power series of ϵ;

$$h^{\mu\nu} = \sum_{n=0} \epsilon^{4+n} {}_{n+4}h^{\mu\nu}.$$

Using this equation, we expand $t^{\mu\nu}_{LL}$ in ϵ.

Here we only show ${}_{10}[-16\pi g t^{\mu\nu}_{LL}]$. See paper II for ${}_{\le 9}[-16\pi g t^{\mu\nu}_{LL}]$. Note that all the divergence such as $h^{\mu k}{}_{,k}$ in paper II should be replaced by $-h^{\mu\tau}{}_{,\tau}$ consistent with the following results. This is simply because it is practically much easier to use $h^{\mu\tau}{}_{,\tau}$ than $-h^{\mu k}{}_{,k}$,

$$\begin{aligned}&{}_{10}[-16\pi g t^{\tau\tau}_{LL}] \\ &= -\frac{7}{4} {}_4h^{\tau\tau}{}_{,k} {}_8h^{\tau\tau,k} + \frac{1}{4} {}_4h^{\tau\tau,l} {}_6h^k{}_{k,l} - {}_4h^{\tau\tau}{}_{,k} {}_6h^{\tau k}{}_{,\tau} + 2 {}_4h^{\tau k,l} {}_6h^\tau{}_{(k,l)} \\ &\quad -\frac{3}{4} {}_4h^{\tau\tau}{}_{,\tau} {}_6h^{\tau\tau}{}_{,\tau} + \frac{7}{4} {}_4h^{\tau\tau} {}_4h^{\tau\tau}{}_{,k} {}_6h^{\tau\tau,k} - {}_4h^{\tau k}{}_{,\tau} {}_6h^{\tau\tau}{}_{,k} + \frac{1}{4} {}_4h^k{}_{k,l} {}_6h^{\tau\tau,l} \\ &\quad -\frac{7}{8} {}_6h^{\tau\tau}{}_{,k} {}_6h^{\tau\tau,k} + \frac{1}{4} {}_4h^{\tau\tau}{}_{,\tau} {}_4h^k{}_{k,\tau} + {}_4h^{\tau k,l} {}_4h_{kl,\tau} \\ &\quad +\frac{1}{4} {}_4h^{kl,m} {}_4h_{kl,m} - \frac{1}{2} {}_4h^{kl,m} {}_4h_{km,l} - \frac{1}{8} {}_4h^k{}_{k,m} {}_4h^l{}_l{}^{,m} \\ &\quad +\frac{1}{4} {}_4h^{\tau k} {}_4h^{\tau\tau}{}_{,\tau} {}_4h^{\tau\tau}{}_{,k} + \frac{7}{8} {}_4h^{kl} {}_4h^{\tau\tau}{}_{,k} {}_4h^{\tau\tau}{}_{,l} - \frac{7}{8} ({}_4h^{\tau\tau})^2 {}_4h^{\tau\tau}{}_{,k} {}_4h^{\tau\tau,k} + \frac{7}{8} {}_6h^{\tau\tau} {}_4h^{\tau\tau}{}_{,k} {}_4h^{\tau\tau,k} \\ &\quad -2 {}_4h^{\tau k} {}_4h^{\tau\tau}{}_{,l} {}_4h^{\tau l}{}_{,k} - \frac{3}{2} {}_4h^{\tau k} {}_4h^{\tau\tau,l} {}_4h^\tau{}_{k,l},\end{aligned} \tag{F2}$$

$$
\begin{aligned}
&{}_{10}[-16\pi g t_{LL}^{\tau i}] \\
&= 2{}_4h^{\tau\tau}{}_{,k8}h^{\tau[k,i]} + \frac{3}{4}{}_4h^{\tau\tau,i}{}_8h^{\tau\tau}{}_{,\tau} + 2{}_4h^{\tau[k,i]}{}_8h^{\tau\tau}{}_{,k} + \frac{3}{4}{}_4h^{\tau\tau}{}_{,\tau 8}h^{\tau\tau,i} \\
&\quad -\frac{1}{4}{}_4h^{\tau\tau,i}{}_6h^{k}{}_{k,\tau} - \frac{1}{4}{}_4h^{\tau\tau}{}_{,\tau 6}h^{k}{}_{k}{}^{,i} + 2{}_4h^{\tau}{}_{k,l6}h^{k[i,l]} \\
&\quad -{}_4h^{\tau i}{}_{,k6}h^{\tau k}{}_{,\tau} - {}_4h^{\tau k}{}_{,\tau 6}h^{\tau i}{}_{,k} + 2{}_4h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k6}h^{\tau[i,k]} \\
&\quad +2{}_4h^{k[i,l]}{}_6h^{\tau}{}_{k,l} + 2{}_6h^{\tau\tau}{}_{,k6}h^{\tau[k,i]} - \frac{3}{4}{}_4h^{\tau\tau}{}_4h^{\tau\tau,i}{}_6h^{\tau\tau}{}_{,\tau} - \frac{1}{4}{}_4h^{k}{}_{k}{}^{,i}{}_6h^{\tau\tau}{}_{,\tau} \\
&\quad +\frac{3}{4}{}_6h^{\tau\tau,i}{}_6h^{\tau\tau}{}_{,\tau} \\
&\quad -\frac{3}{4}{}_4h^{\tau\tau}{}_4h^{\tau\tau}{}_{,\tau 6}h^{\tau\tau,i} + \frac{1}{4}{}_4h^{\tau i}{}_4h^{\tau\tau,k}{}_6h^{\tau\tau}{}_{,k} - \frac{1}{2}{}_4h^{\tau}{}_{k4}h^{\tau\tau,(i}{}_6h^{|\tau\tau|,k)} + 2{}_4h^{\tau\tau}{}_4h^{\tau[i,k]}{}_6h^{\tau\tau}{}_{,k} \\
&\quad -\frac{1}{4}{}_4h^{k}{}_{k,\tau 6}h^{\tau\tau,i} - \frac{1}{2}{}_4h^{kl,i}{}_4h_{kl,\tau} + {}_4h^{ki,l}{}_4h_{kl,\tau} + \frac{1}{4}{}_4h^{k}{}_{k}{}^{,i}{}_4h^{l}{}_{l,\tau} \\
&\quad -\frac{3}{8}{}_4h^{\tau i}({}_4h^{\tau\tau}{}_{,\tau})^2 - \frac{3}{4}{}_4h^{ik}{}_4h^{\tau\tau}{}_{,\tau 4}h^{\tau\tau}{}_{,k} + \frac{3}{4}({}_4h^{\tau\tau})^2{}_4h^{\tau\tau}{}_{,\tau 4}h^{\tau\tau,i} - \frac{3}{4}{}_6h^{\tau\tau}{}_4h^{\tau\tau}{}_{,\tau 4}h^{\tau\tau,i} \\
&\quad -\frac{1}{4}{}_4h^{\tau i}{}_4h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k4}h^{\tau\tau,k} + \frac{1}{8}{}_6h^{\tau i}{}_4h^{\tau\tau}{}_{,k4}h^{\tau\tau,k} + \frac{1}{2}{}_4h^{\tau\tau}{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,k4}h^{\tau\tau,i} \\
&\quad -\frac{1}{4}{}_6h^{\tau k}{}_4h^{\tau\tau}{}_{,k4}h^{\tau\tau,i} + \frac{1}{2}{}_4h^{\tau k}{}_4h^{\tau}{}_{k,\tau 4}h^{\tau\tau,i} - {}_4h^{ik}{}_4h^{\tau\tau}{}_{,l4}h^{\tau l}{}_{,k} + 2({}_4h^{\tau\tau})^2{}_4h^{\tau\tau}{}_{,k4}h^{\tau[k,i]} \\
&\quad -2{}_6h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k4}h^{\tau[k,i]} + {}_4h^{\tau i}{}_4h^{\tau k,l}{}_4h^{\tau}{}_{[l,k]} + \frac{1}{2}{}_4h^{\tau}{}_{k4}h^{\tau\tau}{}_{,\tau 4}h^{\tau k,i} \\
&\quad +2{}_4h^{\tau k}{}_4h^{\tau}{}_{k,l4}h^{\tau[l,i]} + 2{}_4h^{\tau k}{}_4h^{\tau}{}_{l,k4}h^{\tau[l,i]} + {}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,\tau 4}h^{\tau i}{}_{,k} + {}_4h^{kl}{}_4h^{\tau\tau}{}_{,k4}h^{\tau i}{}_{,l} \\
&\quad +\frac{1}{4}{}_4h^{\tau i}{}_4h^{\tau\tau,k}{}_4h^{l}{}_{l,k} - \frac{1}{2}{}_4h^{\tau}{}_{k4}h^{l}{}_{l}{}^{(,k}{}_4h^{|\tau\tau|,i)} + 2{}_4h^{\tau}{}_{k4}h^{\tau\tau}{}_{,l4}h^{k[l,i]},
\end{aligned}
\tag{F3}
$$

$$
\begin{aligned}
&{}_{10}[-16\pi g t^{ij}_{LL}] \\
&= \frac{1}{4}(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})\Big\{{}_4h^{\tau\tau,k}({}_{10}h^{\tau\tau,l} + {}_8h^m{}_m{}^{,l} + 4{}_8h^{\tau l}{}_{,\tau}) + 8{}_4h^\tau{}_m{}^{,k}{}_8h^{\tau[l,m]}\Big\} \\
&\quad +2{}_4h^{\tau i}{}_{,k}{}_8h^{\tau[k,j]} + 2{}_4h^{\tau j}{}_{,k}{}_8h^{\tau[k,i]} - \frac{3}{4}\delta^{ij}{}_4h^{\tau\tau}{}_{,\tau}{}_8h^{\tau\tau}{}_{,\tau} \\
&\quad +\frac{1}{4}(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})\left({}_6h^{\tau\tau,k} + {}_4h^m{}_m{}^{,k} - 2{}_4h^{\tau\tau}{}_4h^{\tau\tau,k} + 4{}_4h^{\tau k}{}_{,\tau}\right){}_8h^{\tau\tau,l} \\
&\quad +\frac{1}{4}\delta^{ij}{}_4h^{\tau\tau}{}_{,\tau}{}_6h^k{}_{k,\tau} - (\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl}){}_4h^{\tau m,k}{}_6h^l{}_{m,\tau} \\
&\quad +\frac{1}{4}(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})\left({}_6h^{\tau\tau,k} - {}_4h^m{}_m{}^{,k} - {}_4h^{\tau\tau}{}_4h^{\tau\tau,k}\right){}_6h^n{}_n{}^{,l} \\
&\quad +(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})\left(\frac{1}{2}{}_4h^{mn,k}{}_6h_{mn}{}^{,l} - {}_4h^{mk,n}{}_6h_{mn}{}^{,l}\right) \\
&\quad +2{}_4h^{k[i,l]}{}_6h^j{}_{k,l} + 2{}_4h^{k[j,l]}{}_6h^i{}_{k,l} \\
&\quad +(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})({}_6h^{\tau\tau,k}{}_6h^{\tau l}{}_{,\tau} - {}_4h^{\tau\tau}{}_4h^{\tau\tau,k}{}_6h^{\tau l}{}_{,\tau}) + 2{}_4h^{\tau(i}{}_{,|\tau|}{}_6h^{j)\tau}{}_{,\tau} \\
&\quad -(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl}){}_4h^k{}_{m,\tau}{}_6h^{\tau m,l} \\
&\quad +(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})\left({}_4h^{\tau\tau}{}_4h^\tau{}_m{}^{,k}{}_6h^{\tau m,l} - {}_4h^{\tau\tau}{}_4h^{\tau k}{}_{,m}{}_6h^{\tau m,l} + \frac{1}{2}{}_4h^\tau{}_m{}_4h^{\tau\tau,k}{}_6h^{\tau m,l}\right) \\
&\quad +2{}_4h^{\tau\tau}{}_4h^{\tau[i,k]}{}_6h^{\tau j}{}_{,k} + 2{}_4h^{\tau\tau}{}_4h^{\tau[j,k]}{}_6h^{\tau i}{}_{,k} \\
&\quad +\frac{1}{8}\delta^{ij}{}_6h^{kl}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau}{}_{,l} + \frac{1}{8}{}_6h^{ij}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,k} - \frac{1}{2}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,(i}{}_6h^{j)k} \\
&\quad +\frac{1}{4}(\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l - \delta^{ij}\delta_{kl})(2{}_6h^\tau{}_m{}_4h^{\tau\tau,k}{}_4h^{\tau m,l} - {}_6h^{\tau k}{}_4h^{\tau\tau}{}_{,\tau}{}_4h^{\tau\tau,l}) \\
&\quad +\delta^{ij}{}_6h^{\tau k,l}{}_6h^\tau{}_{[k,l]} + 2{}_6h^{\tau k,(i}{}_6h^{j)\tau}{}_{,k} - {}_6h^{\tau k,i}{}_6h^\tau{}_k{}^{,j} - {}_6h^{\tau i,k}{}_6h^{\tau j}{}_{,k} \\
&\quad +\frac{1}{4}\delta^{ij}\left(2{}_4h_{kl}{}_4h^{\tau k,m}{}_4h^{\tau l}{}_{,m} - 2{}_4h^{kl}{}_4h^{\tau m}{}_{,k}{}_4h^\tau{}_{m,l} + {}_4h^{kl}{}_4h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau}{}_{,l} - {}_4h^{\tau\tau}{}_4h^{kl}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau}{}_{,l}\right) \\
&\quad +\frac{3}{8}{}_4h^{ij}({}_4h^{\tau\tau}{}_{,\tau})^2 - \frac{1}{4}{}_4h^{\tau\tau}{}_4h^{ij}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,k} + {}_4h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,(i}{}_4h^{j)k} + {}_4h^{ij}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau k}{}_{,\tau} \\
&\quad -{}_4h^{ij}{}_4h^{\tau k,l}{}_4h^\tau{}_{[k,l]} + {}_4h^{kl}{}_4h^{\tau i}{}_{,k}{}_4h^{\tau j}{}_{,l} - {}_4h_{kl}{}_4h^{\tau k,i}{}_4h^{\tau l,j} \\
&\quad +2{}_4h^{ki}{}_4h^{\tau[l,j]}{}_4h^\tau{}_{l,k} + 2{}_4h^{kj}{}_4h^{\tau[l,i]}{}_4h^\tau{}_{l,k} - 2{}_4h^{k(i}{}_4h^{j)\tau}{}_{,\tau}{}_4h^{\tau\tau}{}_{,k} \\
&\quad +\frac{1}{4}{}_4h^{ij}{}_4h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau,k} - \frac{1}{2}{}_4h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau,(i}{}_4h^{j)k} - \frac{1}{2}{}_6h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,(i}{}_4h^{j)k} \\
&\quad +\delta^{ij}\Bigg(2{}_4h^\tau{}_k{}_4h^\tau{}_{l,m}{}_4h^{k[l,m]} - \frac{1}{2}{}_4h^\tau{}_k{}_4h^{\tau k,l}{}_4h^m{}_{m,l} - \frac{1}{4}({}_4h^{\tau\tau})^2{}_4h^{\tau\tau,k}{}_4h^l{}_{l,k} \\
&\quad +\frac{1}{4}{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,\tau}{}_4h^l{}_{l,k}\Bigg) \\
&\quad +\delta^{ij}\left(\frac{1}{4}{}_4h^{\tau\tau}{}_4h^l{}_{l,k}{}_6h^{\tau\tau,k} + \frac{1}{4}{}_6h^{\tau\tau}{}_4h^l{}_{l,k}{}_4h^{\tau\tau,k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}{}_4h^{\tau\tau}{}_{,\tau}{}_4h^k{}_k{}^{,(i}{}_4h^{j)\tau}+\frac{1}{2}({}_4h^{\tau\tau})^2{}_4h^k{}_k{}^{,(i}{}_4h^{|\tau\tau|,j)}-\frac{1}{2}{}_6h^{\tau\tau}{}_4h^k{}_k{}^{,(i}{}_4h^{|\tau\tau|,j)}\\
&+{}_4h^\tau{}_k{}_4h^l{}_l{}^{,(i}{}_4h^{|\tau k|,j)}+2{}_4h^{\tau k}{}_4h^{\tau[i,l]}{}_4h_{kl}{}^{,j}+2{}_4h^{\tau k}{}_4h^{\tau[j,l]}{}_4h_{kl}{}^{,i}\\
&+2{}_4h^{\tau k}{}_4h^{\tau[l,i]}{}_4h^j{}_{k,l}+2{}_4h^{\tau k}{}_4h^{\tau[l,j]}{}_4h^i{}_{k,l}-\frac{1}{2}{}_4h^{\tau\tau}{}_4h^k{}_k{}^{,(i}{}_6h^{|\tau\tau|,j)}\\
&+(\delta^i{}_k\delta^j{}_l+\delta^j{}_k\delta^i{}_l-\delta^{ij}\delta_{kl})({}_4h^\tau{}_m{}_4h^{\tau\tau,k}{}_4h^{ml}{}_{,\tau}-\frac{1}{4}{}_4h^{\tau k}{}_4h^{\tau\tau,l}{}_4h^m{}_{m,\tau})\\
&+\delta^{ij}\Bigg(-\frac{3}{8}({}_4h^{\tau\tau})^2({}_4h^{\tau\tau}{}_{,\tau})^2+\frac{3}{8}{}_6h^{\tau\tau}({}_4h^{\tau\tau}{}_{,\tau})^2+\frac{1}{2}({}_4h^{\tau\tau})^3{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,k}-\frac{1}{2}{}_4h^{\tau\tau}{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,\tau}{}_4h^{\tau\tau}{}_{,k}\\
&-\frac{3}{4}{}_4h^{\tau\tau}{}_6h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,k}+\frac{1}{4}{}_8h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k}{}_4h^{\tau\tau,k}+\frac{1}{4}{}_4h^\tau{}_k{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,l}{}_4h^{\tau\tau,l}-\frac{1}{2}{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,\tau}{}_4h^\tau{}_{k,\tau}\\
&-({}_4h^{\tau\tau})^2{}_4h^{\tau\tau,k}{}_4h^\tau{}_{k,\tau}+{}_6h^{\tau\tau}{}_4h^{\tau\tau,k}{}_4h^\tau{}_{k,\tau}+{}_4h^{\tau\tau}{}_4h^{\tau k}{}_4h^{\tau\tau,l}{}_4h^\tau{}_{k,l}-{}_4h^\tau{}_k{}_4h^{\tau k}{}_{,l}{}_4h^{\tau l}{}_{,\tau}\\
&+({}_4h^{\tau\tau})^2{}_4h^{\tau k,l}{}_4h^\tau{}_{[k,l]}-{}_6h^{\tau\tau}{}_4h^{\tau k,l}{}_4h^\tau{}_{[k,l]}-{}_4h^{\tau k}{}_4h^\tau{}_{l,k}{}_4h^{\tau l}{}_{,\tau}+\frac{1}{4}{}_4h^k{}_{l,\tau}{}_4h^l{}_{k,\tau}\\
&+\frac{1}{4}{}_4h^{kl}{}_4h^{\tau\tau}{}_{,k}{}_4h^m{}_{m,l}-\frac{1}{8}{}_4h^k{}_{k,\tau}{}_4h^l{}_{l,\tau}+\frac{1}{4}{}_4h^k{}_{k,\tau}{}_6h^{\tau\tau}{}_{,\tau}-\frac{1}{4}{}_4h^{kl}{}_4h^{\tau\tau,m}{}_4h_{kl,m}+\frac{3}{4}{}_4h^{\tau\tau}{}_4h^{\tau\tau}{}_{,\tau}{}_6h^{\tau\tau}{}_{,\tau}\\
&+\frac{1}{4}{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau}{}_{,\tau}-\frac{3}{8}({}_6h^{\tau\tau}{}_{,\tau})^2+\frac{1}{4}{}_4h^{\tau k}{}_4h^{\tau\tau}{}_{,\tau}{}_6h^{\tau\tau}{}_{,k}+{}_4h^{\tau\tau}{}_4h^{\tau k}{}_{,\tau}{}_6h^{\tau\tau}{}_{,k}\\
&-\frac{1}{2}{}_4h^{\tau k}{}_4h^\tau{}_{k,l}{}_6h^{\tau\tau,l}-\frac{3}{4}({}_4h^{\tau\tau})^2{}_4h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau,k}+\frac{1}{2}{}_6h^{\tau\tau}{}_4h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau,k}+\frac{1}{4}{}_4h^{\tau\tau}{}_6h^{\tau\tau}{}_{,k}{}_6h^{\tau\tau,k}\Bigg)\\
&+{}_4h^{\tau\tau}{}_4h^{\tau\tau}{}_{,\tau}{}_4h^{\tau\tau,(i}{}_4h^{j)\tau}-({}_4h^{\tau\tau})^3{}_4h^{\tau\tau,i}{}_4h^{\tau\tau,j}-\frac{1}{2}{}_4h^{\tau k}{}_4h^\tau{}_k{}_4h^{\tau\tau,i}{}_4h^{\tau\tau,j}\\
&+\frac{3}{2}{}_4h^{\tau\tau}{}_6h^{\tau\tau}{}_4h^{\tau\tau,i}{}_4h^{\tau\tau,j}-\frac{1}{2}{}_8h^{\tau\tau}{}_4h^{\tau\tau,i}{}_4h^{\tau\tau,j}\\
&-2{}_4h^{\tau\tau}{}_4h^\tau{}_k{}_4h^{\tau\tau,(i}{}_4h^{|\tau k|,j)}+2{}_4h^\tau{}_{k,\tau}{}_4h^{\tau k,(i}{}_4h^{j)\tau}\\
&+2({}_4h^{\tau\tau})^2{}_4h^\tau{}_k{}^{,j}{}_4h^{\tau[i,k]}-2({}_4h^{\tau\tau})^2{}_4h^{\tau j}{}_{,k}{}_4h^{\tau[i,k]}+2({}_4h^{\tau\tau})^2{}_4h^{\tau\tau,(i}{}_4h^{j)\tau}{}_{,\tau}\\
&-2{}_6h^{\tau\tau}{}_4h^\tau{}_k{}^{,j}{}_4h^{\tau[i,k]}+2{}_6h^{\tau\tau}{}_4h^{\tau j}{}_{,k}{}_4h^{\tau[i,k]}-2{}_6h^{\tau\tau}{}_4h^{\tau\tau,(i}{}_4h^{j)\tau}{}_{,\tau}\\
&-2{}_4h^{\tau\tau}{}_{,\tau}{}_4h^{\tau(i}{}_4h^{j)\tau}{}_{,\tau}+2{}_4h^\tau{}_k{}_4h^{\tau k,(i}{}_4h^{j)\tau}{}_{,\tau}-2{}_4h^{\tau k}{}_{,\tau}{}_4h^{\tau(i}{}_4h^{j)\tau}{}_{,k}+2{}_4h^\tau{}_k{}_4h^{\tau(i,|k|}{}_4h^{j)\tau}{}_{,\tau}\\
&+\frac{1}{4}{}_4h^{ij}{}_4h^{\tau\tau,k}{}_4h^l{}_{l,k}-\frac{1}{2}{}_4h^l{}_{l,k}{}_4h^{\tau\tau,(i}{}_4h^{j)k}-\frac{1}{2}{}_4h^{\tau\tau}{}_{,k}{}_4h^l{}_l{}^{,(i}{}_4h^{j)k}+\frac{1}{2}{}_4h_{kl}{}_4h^{\tau\tau,(i}{}_4h^{|kl|,j)}\\
&-{}_4h^{ik}{}_{,\tau}{}_4h^j{}_{k,\tau}-\frac{1}{2}{}_6h^{\tau\tau}{}_{,\tau}{}_4h^{\tau\tau,(i}{}_4h^{j)\tau}-\frac{1}{2}{}_4h^{\tau\tau}{}_{,\tau}{}_6h^{\tau\tau,(i}{}_4h^{j)\tau}+\frac{3}{2}({}_4h^{\tau\tau})^2{}_4h^{\tau\tau,(i}{}_6h^{|\tau\tau|,j)}\\
&-{}_6h^{\tau\tau}{}_6h^{\tau\tau,(i}{}_4h^{|\tau\tau|,j)}+{}_4h^\tau{}_k{}_4h^{\tau k,(i}{}_6h^{|\tau\tau|,j)}-2{}_4h^{\tau\tau}{}_6h^{\tau\tau,(i}{}_4h^{j)\tau}{}_{,\tau}-\frac{1}{2}{}_4h^{\tau\tau}{}_6h^{\tau\tau,i}{}_6h^{\tau\tau,j}.
\end{aligned}
\tag{F4}
$$

APPENDIX G: SUPERPOTENTIALS

Here we list useful superpotentials. Below, $f^{(m,n)}$ satisfies $\Delta f^{(m,n)} = r_1^m r_2^n$. Similarly, $\Delta f^{(m;\ln,n;\ln)} = r_1^m r_2^n \ln r_1 \ln r_2$. The appendix in [33] gives a greatly useful list of superpotentials including $f^{(\ln S)}$, $f^{(1,1)}$, $f^{(-1,1)}$, and $f^{(3,-1)}$. Other useful superpotentials are given in [28,40],

$$
\begin{aligned}
f^{(-2,2)} &= \frac{r_1^2}{6} + \frac{1}{3}(-r_1^2 + 2r_{12}^2 + r_2^2)\ln r_1, \\
f^{(-3,-3)} &= \frac{1}{r_{12}^3 r_1}\ln\left(\frac{S}{r_1}\right) + \frac{1}{r_{12}^3 r_2}\ln\left(\frac{S}{r_2}\right) - \frac{1}{r_{12}^2 r_1 r_2}, \\
f^{(-3,-2)} &= \frac{1}{r_1 r_{12}^2}\ln\left(\frac{r_2}{r_1}\right), \\
f^{(-3,-1)} &= \frac{1}{r_1 r_{12}}\ln\left(\frac{S}{r_1}\right), \\
f^{(-3,0)} &= \frac{-\ln r_1}{r_1}, \\
f^{(-3,1)} &= -\frac{r_2}{r_1} + \ln S + \frac{r_{12}}{r_1}\ln\left(\frac{S}{r_1}\right), \\
f^{(-4,-3)} &= \frac{1}{2}\Delta_{11} f^{(-2,-3)}, \\
f^{(-4,-1)} &= \frac{r_2}{2r_1^2 r_{12}^2}, \\
f^{(0;\ln,0)} &= \frac{1}{6}r_1^2 \ln r_1 - \frac{5}{36}r_1^2.
\end{aligned}
$$

An example below shows how we construct superpotentials from simpler ones,

$$
\begin{aligned}
f^{(6,-2)} = {} & -6f^{(0;0,4;\ln)} - 3f^{(2,2)} + \frac{24r_{12}^2}{7}f^{(0;0,2;\ln)} - \frac{24r_{12}^2}{7}f^{(2;0,0;\ln)} + \frac{6r_{12}^2}{7}f^{(4,-2)} \\
& + \frac{r_1^6}{14} - \frac{3}{35}r_{12}^2 r_1^4 + \frac{3}{70}r_{12}^2 r_2^4 - \frac{r_2^6}{49} + \frac{r_1^6}{7}\ln r_2 - \frac{3}{7}r_1^4 r_2^2 \ln r_2 + \frac{3}{7}r_1^2 r_2^4 \ln r_2,
\end{aligned}
$$

which is computed from $f^{(0;0,4;\ln)}$, $f^{(2,2)}$, $f^{(0;0,2;\ln)}$, $f^{(2;0,0;\ln)}$, and $f^{(4,-2)}$. They are

$$
\begin{aligned}
f^{(0;0,4;\ln)} &= \frac{1}{42}r_2^6 \ln r_2 - \frac{13}{1764}r_2^6, \\
f^{(2,2)} &= -\frac{r_1^6}{84} + \frac{1}{70}r_1^4 r_{12}^2 + \frac{1}{28}r_1^4 r_2^2, \\
f^{(0;0,2;\ln)} &= \frac{1}{20}r_2^4 \ln r_2 - \frac{9}{400}r_2^4, \\
f^{(2;0,0;\ln)} &= -\frac{7}{200}r_1^4 + \frac{1}{60}r_1^2 r_{12}^2 - \frac{7}{180}r_{12}^2 r_2^2 + \frac{1}{80}r_2^4 + \frac{1}{10}r_1^2 r_2^2 \ln r_2 + \frac{1}{15}r_{12}^2 r_2^2 \ln r_2 - \frac{1}{20}r_2^4 \ln r_2, \\
f^{(4,-2)} &= \frac{1}{5}r_1^4 \ln r_2 - \frac{1}{25}r_1^4 + \frac{4}{5}r_{12}^2 f^{(2,-2)} - 4f^{(2;0,0;\ln)}.
\end{aligned}
$$

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